Implementing the Optimal Auction*

Bernard Caillaud[†] and Jacques Robert[‡]

This version October 2000

*We have benefited from comments by seminar participants at the Econometric Society North American Winter Meeting (New Orleans) and European Meeting (Toulouse), PURC University of Florida, University of Wisconsin. The authors also wish to acknowledge the hospitality of IDEI in Toulouse where they started this research. The second author acknowledges the financial support of FCAR, CRSHC and CIRANO.

[†] CERAS-ENPC (CNRS), 28 rue des Saints-Pères, 75007 Paris, France, and CEPR, London, UK. E-mail: caillaud@enpc.fr.

[‡] Dép. de Sciences Économiques, Université de Montréal, CRDE et CIRANO, C.P.6128 Centre-ville, Montréal, QC, Canada H3C 3J7, E-mail:robertj@cirano.umontreal.ca.

Proposed Running Header: Implementing the Optimal Auction

Correspondence to: Professor Bernard Caillaud, CERAS - ENPC, 28 rue des Saints-Pères, 75007 Paris, France; email:caillaud@enpc.fr.

ABSTRACT:

In a general framework with independent private values of the bidders, we start from a simple second-price auction that implements the optimal auction, where the winner pays a pre-specified price conditional on the winning bid. We then propose a modified game that allows to implement the optimal auction outcome *when the seller ignores the distributions of the different bidders' valuations.* In this robust implementation procedure, a second-price auction is organized and the winner volunteers a payment to the seller; this payment can then be challenged by another bidder who knows the distribution of the winner's valuation. We finally propose refinements of this simple procedure that ensure uniqueness of the resulting equilibrium.

1 Introduction

Since the late seventies, the mechanism design literature has been much successful is determining the form and properties of desirable institutions in situations where informational problems arise.[1] As a leading example, Myerson [1981] characterizes the revenue-maximizing auction when potential buyers have private and independent valuations for the good on sale and are risk-neutral. In the simplest case, the revenue-maximizing auction can be implemented using a simple first-price auction (alternatively, a second-price or an ascending English auction will do as well since the revenue equivalence applies) with the appropriate reserve price.

The mechanism design approach has however been criticized on the following grounds. First, optimal institutions as derived by this approach are often much more complex than real-life institutions. For example, the optimal auction mechanism turns out to be much more complicated than simple first or second-price auctions with ex ante asymmetric participants (or in non-regular cases). Second and most importantly, the mechanism design approach is said to be *informationally demanding:* even when optimal institutions turn out to be simple enough, their design requires an unrealistic degree of knowledge concerning details of the economic environment. For example, the optimal auction in the regular and symmetric case requires the appropriate choice of the reserve price, which strongly depends upon the knowledge of the prior distributions of tastes in the population of potential bidders. In repeated environments, simulation-based estimation methods can help figure out the objective distribution of tastes in a stable population,[2] although

[1] See e.g. the recent experience of PCS auctions (Cramton [1995], McAfee-McMillan [1996]).

[2] See among others Laffont-Ossard-Vuong [1995] and Donald-Paarsch [1996]. McAfee-Quan-Vincent [1996] uses econometric estimations to calculate the optimal reserve prices in housing auctions.

they usually miss the strategic dimension that is precisely due to this repeated-game setting. But for unusual auction situations (defense procurement, auction for monopoly franchises...) the crucial data concern the agents' subjective beliefs on the distribution of tastes, which are far more difficult to measure. The spirit of the so-called Wilson Doctrine has then been to ask for robustness, that is to try to reach detail-free conclusions within the mechanism design approach.

In this paper, we take these critiques seriously and we look for robust implementation procedures for the optimal auction in Myerson's framework. We propose and analyze a game that can be easily organized by an ignorant seller, who has no information not only on the bidders' valuations but also on the prior distributions of these valuations. This game can be put easily in practice and it relies on competition between rival bidders who may challenge the potential winner on the basis of the price he proposes to pay to the seller. We finally discuss several refinements so as to guarantee uniqueness of the equilibrium that emerges in such a procedure.

In section 2, we first introduce a simple pedagogical way to implement and interpret the optimal revenue-maximizing auction, directly inspired by Bulow-Roberts [1989]. In this game, an ascending-price auction is organized; the winner then pays to the seller a pre-specified price conditional on the winning bid of the ascending auction. In the spirit of Bulow-Roberts [1989], this price can be viewed as a monopolistic pricing decision against the winner of the auction, where the monopolist's cost is determined by the winning bid in the auction. In order to do so, the seller needs to be able to compute the optimal monopoly price for each participant.

We then assume that the auction designer has little or no knowledge on the distribution of private information, the structure of preferences, etc. Hence, we assume the seller is ignorant and we look for a simple mechanism which is independent of the parameters of the problem but nevertheless achieves the maximal revenue for the seller. We simply assume that it is common knowledge that bidders $j \neq i$ share the same beliefs on bidder i's valuation.[3] Formally, we are looking for *game forms that are independent of the distribution of types* and which induce as an equilibrium the desired optimal allocation: we call this a *universal implementation procedure* of the optimal auction.

The game we propose in section 3 and 4 has the following relatively simple structure: (i) an ascending-price auction is first organized and the winner and the winning bid are made public information; (ii) the winner of this auction then volunteers a payment to the seller that is also publicly disclosed; (iii) another participant is then designated randomly or through a secondary auction and he can then challenge the price proposed by making a take-it-or-leave-it offer at a higher price to the ascending-auction winner. This game admits an equilibrium that generates, for all distributions of bidders' valuations, the revenue-maximizing auction outcome. In this equilibrium, instead of charging a monopolistic price against the winner, the seller delegates this right to another bidder who has better information to solve the monopolistic pricing problem. The threat of having to face this monopoly price anyway induces the winner of the ascending auction to volunteer this payment, so as to avoid a fee involved in case of a challenge.

Our game admits in fact several perfect Bayesian equilibria that we completely characterize, but the implementation equilibrium has distinctive properties that make it "focal". We develop this argument in section 5 along two lines that allow us to get unique implementation of the optimal auction outcome: we propose a small

[3] Our strong results enable us to dispense with an explicit Bayesian setting where bidders' private information not only concerns their own valuation but also their first-degree beliefs on each others' valuations (see Harsanyi [1968] and Mertens-Zamir [1985]).

perturbation of the extensive-form game in the spirit of Fudenberg-Kreps-Levine [1988] such that the implementation equilibrium is the unique limit equilibrium when the perturbation vanishes; and, in the unperturbed game, we investigate a restriction on beliefs due to Reny [1992] which again selects the implementation equilibrium as the unique reasonable or "explicable" equilibrium.

2 The IPV model and the optimal auction

We consider the classical auction setting with Independent Private Values (IPV), as analyzed in Myerson [1981]. A risk-neutral seller wants to maximize her revenue from the sale of an indivisible good for which her valuation is known and normalized to 0. Ther are n risk-neutral potential buyers, each with private information on his own valuation v_i for the good. Valuations v_i, $i = 1, ...n$ are independently drawn from continuously differentiable distributions $F_i(.)$, with densities $f_i(.)$ and full support $[\underline{v}_i, \bar{v}_i]$. These distributions are common knowledge among all agents.

Before addressing the auction design problem, let first consider the corresponding pricing problem of a monopolist with unit cost b facing demand $[1 - F_i(p)]$. Let first $P_i(b)$ denote the set of optimal monopoly prices:

$$P_i(b) \equiv \arg\max_p \; \{(p - b)[1 - F_i(p)]\}.$$

Then, let $t_i(b)$ and $r_i(b)$ denote respectively the lowest and the largest elements in $P_i(b)$. Accordingly, we define the inverse of the convex hull of $P_i(b)$: we let $H_i(v_i)$ be the (unique) value b such that $v_i \in [t_i(b), r_i(b)]$.[4]

[4] $P_i(b)$ maximizes the continuous function $[(p - b)(1 - F_i(p)]$ over the support $[\underline{v}_i, \overline{v}_i]$. Thus, it has closed graph and is non-empty, and $t_i(b)$ and $r_i(b)$ are well-defined. Furthermore, under the assumption that $F_i(.)$ is continuously differentiable, one can show that if $b < b'$, then $r_i(b) < t_i(b')$

The function $H_i(.)$ corresponds to the well-known virtual valuation function in auction design theory. Indeed, whenever the monopolist's pricing problem is well-behaved, i.e. whenever $P_i(b)$ is single-valued and invertible, we have: $H_i(v_i) \equiv \{b|v_i = P_i(b)\} = J_i(v_i) \equiv v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ which corresponds to the virtual valuation function in the so-called *regular case.* Whenever the function $J_i(v_i) \equiv v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is not monotone, the virtual valuation function is obtained by ironing out the function $J_i(v_i)$. One can verify that this yields the function $H_i(v_i)$.

Given the above notation, the optimal revenue-maximizing auction requires to transfer the good to bidder i with valuation v_i if:

$$H_i(v_i) \geq \sup\{\sup\{H_h(v_h), h \neq i\}, 0\} \tag{1}$$

that is to the bidder with the highest non-negative "virtual valuation". In case of equality, any tie-breaking rule can be used.

The outcome of the revenue-maximizing auction can be implemented (in dominant strategies) using a direct revelation mechanism, with the appropriate payment function. For example, in the regular case, the winner pays the lowest valuation that would have made him win. Although in the symmetric regular case, the revenue equivalence theorem shows that a first-price, or a second-price, or else an ascending auction, all with reservation price $J^{-1}(0)$, yield the optimal outcome,

and $\lim_{b' \downarrow b} t_i(b') = r_i(b)$. Hence one can verify that for all $v_i \in (\underline{v}_i, \overline{v}_i)$, the expression $H_i(v_i)$ is well-defined, i.e. there is indeed a unique b such that $v_i \in [t_i(b), r_i(b)]$. For extreme values of v_i, we let:

$$H_i(\underline{v}_i) \equiv \sup\{b| \ \underline{v}_i = t_i(b)\}$$
$$H_i(\overline{v}_i) \equiv \inf\{b| \ \overline{v}_i = r_i(b)\}$$

it is difficult in more general (asymmetric, non-regular) cases to find a natural "indirect mechanism", such as a standard auction, that implements the optimal auction.

Bulow-Roberts [1989], however, shows that the seller's problem is formally equivalent to a third-degree price discriminatory monopoly problem with capacity constraint. Facing perfectly identifiable demand functions $[1 - F_i(p)]$ for $i = 1, 2, ...n$, such a monopolist should optimally sell to the buyers with the highest marginal revenue. When the revenue functions $R_i(Q) \equiv QF_i^{-1}(1 - Q)$ are strictly concave, or equivalently when the functions $H_i(.)$ are all strictly increasing, the optimal monopolistic policy is to compare marginal revenues $R_i'([1 - F_i(v_i)])$, that is $H_i(v_i)$ or equivalently $J_i(v_i)$ for $i = 1, ...n$, and allocate the good to the buyer with highest marginal revenue. When the revenue function is not concave, the monopolist should consider the smallest concave upper envelope of $R_i(.)$ and compare marginal revenues computed on this basis, which yields a comparison of the $H_i(v_i)$.

This interpretation of Bulow-Roberts [1998] translates into an alternative implementation procedure of the optimal outcome. Consider the following game:

GAME 1:

1. Bidders participate in an ascending-price auction with initial price starting at zero. The winner is the last participant to drop out (or drawn among the last participants to drop out, with equal probabilities). The "winning bid" is the highest price for which the number of active bidders is larger or equal to 2. If no one participates, the unit is not sold.

2. If i wins by remaining the unique active bidder above the winning bid b, and if exactly m bidders simultaneously drop out at b, bidder i must pay

$p = \frac{1}{m+1}t_i(b) + \frac{m}{m+1}r_i(b)$ to the auctioneer.

3. If i wins as the result of a random draw among several bidders who dropped out exactly at b, bidder i must pay $t_i(b)$ to the auctioneer.

4. Note that in the regular case, cases 2 and 3 both collapse to: $p = P_i(b) = t_i(b) = r_i(b)$.

The key feature of this procedure is that it is a (weakly) dominant strategy for each bidder to bid his virtual valuation. For i of type v_i it is a dominant strategy to stay active whenever b is such that $v_i > r_i(b)$ and drop out whenever $v_i < t_i(b)$, i.e. to stay active until the price reaches the value $H_i(v_i)$. Moreover, one can easily check that the payments implied by Game 1 are identical to the ones specified in Myerson [1981] (page 69, expression (6.8)). We summarize this in the following proposition.

Proposition 1 *: (Bulow-Roberts) Game 1 implements the optimal auction in dominant strategies.*

Game 1 follows a common index procedure, "a common clock": the index increases until only one participant remains, thereby determining the winning bid and the winner. This bid only serves as a basis to determine the actual payment by the winner, based on the winner's virtual valuation function. The final allocation and payment rules can be seen as resulting from a monopoly pricing decision against the winning bidder where the monopolist's cost is determined by the winning bid. This payment could be extracted in a take-it-or leave-it offer game by any intermediary facing a cost equal to the winning bid and knowing the prior distribution of valuations for the winner. We exploit this idea below.

3 Universal implementation in the regular case

The previous implementation game is not informationally demanding for bidders. The dominant strategy implementation procedure does not put strong requirements on the knowledge bidders have upon each others and upon the market conditions as a whole. The auctioneer, however, needs to be able to maximize $(p-b)\,[1-F_i(p)]$ for all i and b. Basically, she needs to know the distributions of taste $F_i(.)$ for each bidder i. Such a requirement may be unrealistic. In many cases, the auctioneer has little knowledge about the parameters of the market, at least compared to the knowledge actual participants in the market have from long years of practice and competition within the market.

We take this critique seriously and look for a "universal" mechanism which can be set up by the auctioneer without such a precise knowledge of the bidders' tastes. We completely reverse the informational requirement by assuming that participants in the auction are better informed than the auction designer upon the distribution of tastes among bidders. More precisely, the mechanism exploits the fact that the distributions $F_i(\cdot)$ are common knowledge among bidders. On the other hand, we do not need specify what are the seller's prior, if any, on these distributions, since the mechanism we present implements the optimal auction whatever the distributions $F_i(.)$. In this mechanism, competition among participants is used as a device to induce participants to implicitly reveal the optimal sale price.

It is important to note there are many mechanisms which can force bidders to reveal to the auctioneer the information necessary to implement the optimal auction. Clearly, selecting among the possible games raises methodological issues. Following the seminal work by Maskin [1977], many studies have been done on

Bayesian Nash implementation.[5] The basic conclusion of this literature is that information which is common knowledge among agents can be revealed at no cost to the principal. One way is to ask agents to reveal simultaneously their joint information, if they fail to send the same information they receive infinite penalties. But the mechanisms proposed are not necessarily "reasonable" or "practical", although we admit these terms are not well defined. We propose an alternative procedure where the seller organizes the auction in a rather conventional fashion and let competitive pressure works in her favor to extract the maximum expected rents. The game is meant to be intuitive and practical.[6]

Let us concentrate on the regular case where $P_i(.)$ is single-valued, i.e. $P_i(b) = t_i(b) = r_i(b)$ (alternatively, all $H_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ are strictly increasing). The proposed game is as follow:

GAME 2

1. Bidders participate in an *ascending-price auction* with initial starting price at zero. The winner i and the winning bid b are determined as in Game 1 and the outcome (i, b) is publicly disclosed.

2. If i wins, he *volunteers a price* p which is publicly disclosed.

3. If i has won at winning price b and has volunteered a price p, another agent is randomly designated and can *challenge* p by proposing a take-it-or-leave-it

[5] See Moore [1992] for a survey on implementation under complete information and Palfrey [1992] for a survey on Bayesian implementation.

[6] Pursuing a similar quest for simple implementation procedures, Glazer-Ma [1989] have analyzed implementation with the possibility of challenges. The procedure we propose also incorporates the possibility of challenges.

offer at a higher price $q > p$, which i can accept or refuse; payments are as follows:

- if p is unchallenged, then i receives the unit and pays p to the seller.
- if a challenge $q > p$ is accepted by i, then i receives the unit, pays a fee Δ to the seller and pays q to the challenger. The challenger pays p to the seller.
- if a challenge $q > p$ is rejected, the seller keeps the good, receives a fee Δ from i and a payment $p - b$ from the challenger.

As in Game 1, a first ascending auction is organized so as to determine the winner and a winning bid. In contrast with Game 1, the winner is then asked to volunteer a payment. The game is designed so that he has an incentive to volunteer the optimal auction payment. He is disciplined in doing so by the possibility of challenges from an informed outside agent, who is able to compute the optimal auction payment based on his knowledge of the prior distribution of the winner's valuation and on the public information about the winner and the winning bid.

To provide some intuition on Game 2, it is worth sketching the equilibrium analysis.

Note first that the first stage winner i should accept a challenge if and only if the challenging price is smaller than his valuation for the good. A potential challenger should then look for the challenge price q that maximizes his expected profit, where the expectation is taken with respect to his posteriors on the winner's type v_i after observing that i wins at winning bid b and volunteers price p. Suppose that, in equilibrium, bidders conjecture that i's strategy in the ascending auction is to drop out at b, when he is of type $v_i = P_i(b)$. The challenger's posteriors are then given by the Bayesian updating of his priors, conditional on the event $\{v_i \geq P_i(b)\}$.[7] In this case, if indeed $b < \bar{v}_i$,[8] the program for the optimal challenging price is given by:

$$\max_{q \geq 0} \left[-(p-b) + (q-b) \inf \left\{ \frac{1-F_i(q)}{1-F_i(P_i(b))}; 1 \right\} \right]. \tag{2}$$

where the infimum corresponds to the challenger's probability assessment that the challenge q will be accepted. The optimum challenge price then corresponds to: $q = P_i(b)$. This is indeed the challenge price if it is strictly larger than p; otherwise, the value of the program above is strictly negative when restricted to $q > p$, and p is not challenged.

Suppose then that bidder i wins at winning bid b. Proposing a price $p < P_i(b)$ generates a challenge $q = P_i(b)$. It cannot be an equilibrium strategy for bidder i to encourage a challenge that he will accept, since he could have immediately proposed $p = P_i(b)$, thereby avoiding to pay the fee Δ. Therefore, the maximal profits that bidder i of type v_i can obtain after winning at winning bid b are equal to: $\sup\{v_i - P_i(b); -\Delta\}$. These profits are non-negative if and only if $b \leq H_i(v_i)$ or $v_i \geq P_i(b)$. It follows that by dropping out in the ascending auction precisely at bid $H_i(v_i)$ and by proposing the unchallengeable price $p = P_i(b)$ when he wins, bidder i of type v_i maximizes his expected gains.

Note finally that the beliefs that have been posited are actually Bayesian consistent. They are deduced from Bayes rule after history (i, b). The fact that p does not induce a further updating is consistent with the fact that among all types

[7]Note that posteriors do not depend upon the price proposal p, although this proposal could serve as a signalling device for the winner.

[8]If $b > \overline{v}_i$, the beliefs proposed in the text are not compatible with the history of the game. One can fix beliefs to be concentrated on $\{v_i = \overline{v}_i\}$, resulting in $q = \overline{v}_i$. The same challenge price should be considered if $b = \overline{v}_i$.

$v_i \geq P_i(b)$, it is a full pooling equilibrium to volunteer the same price $p = P_i(b)$; therefore, p does not convey any additional information about the winner's type.

To summarize, *Game 2 admits a perfect Bayesian equilibrium that implements the optimal auction outcome based on the distributions $F_i(.)$, irrespective on how coarser is the seller's information about the distribution of tastes.* In this equilibrium, bidders bid (adopt a strategy of dropping out at) their virtual valuation $H_i(v_i)$, the winning bid therefore coincides with the highest second virtual valuation. When they win, bidders submit the lowest unchallengeable price to avoid paying the fee Δ; this price corresponds to the highest valuation that would have still enabled them to win the ascending auction, $P_i(b)$, i.e. the corresponding optimal auction payment.

The challenger's program (2) can be viewed as the program of a monopolist facing demand $Q = 1 - F_i(q)$ and unit cost b (for $q \in [P_i(b), \bar{v}_i]$). So, the game relies on the existence of other agents who have priors $F_i(.)$ on i's valuation and can challenge the volunteered payment when it is lower than the corresponding monopoly price. The seller in fact delegates the monopoly pricing decision discussed in section 2 to one better informed intermediary as an off-equilibrium threat that serves as a disciplining device to induce the correct price proposal by the winner. Note that for the system to work, the challenging agent need not be interested in purchasing the good for himself, he need only be motivated by the possibility of a profitable arbitrage. Note also that the challenge could alternatively be organized as a competition game between potential challengers where only the largest challenge price is considered, or as a (common value) auction between potential challengers so as to win the right to make a take-it-or-leave-it offer to the winner of the first auction.

Before turning to the general statement of these results, let us come back on the equilibrium analysis. Take any strictly increasing function $b_i(.)$ such that for all v_i, $b_i(v_i) \leq H_i(v_i)$. Consider the following strategies: bidder i of type v_i drop out in the ascending auction at bid $b_i(v_i)$; if he wins at b, he volunteers a price $p = b_i^{-1}(b)$ (if $b > b_i(\underline{v}_i)$, and $\underline{v}_i$ otherwise) and beliefs on i's type following history (i, b, p) correspond to the updating of priors conditional on the event $\left\{ v_i \geq b_i^{-1}(b) \right\}$. It is easy to see that Program (2) is still valid after replacing $P_i(b)$ by $b_i^{-1}(b)$. Under the regularity assumption and since $P_i(b) \leq b_i^{-1}(b)$, the optimal challenge price is then obtained as a corner solution at $q = b_i^{-1}(b)$ (or $\underline{v}_i$). The argument is then similar to the one developed above.

It follows that the proposed strategies and beliefs constitute another perfect Bayesian equilibrium of Game 2. Game 2 has indeed a continuum of equilibria. In fact, all the equilibria rely on a less aggressive behavior from bidders in the ascending auction compared to the equilibrium that implements the optimal auction; the challenging stage then induces higher prices, for a given winning bid b. Our previous result is therefore a weak implementation result, but we exploit this property of other equilibria to strengthen our points in Section 5.

4 Universal implementation: the general case

In this more technical section, we propose an extension of Game 2 to the general (non-regular) case and we provide a formal statement of our results.

The general game we consider is the following:

GAME 2':

1. Bidders participate in an ascending-price auction as in Game 2, with public disclosure of the winner and the winning bid (i, b).

2. If i wins, he volunteers two prices p_1 and p_2, with $p_2 \geq p_1$. If i dropped out at the winning bid (tie between potential winners), the seller sets $p = p_1$. Otherwise, the seller sets $p = p_1$ with probability $\frac{1}{m+1}$ and $p = p_2$ with probability $\frac{m}{m+1}$, if m bidders simultaneously dropped out at b.

3. The price p is publicly disclosed, as well as whether p is equal to p_1 or p_2. Another agent can then challenge p as in Game 2.

Game 2' extends Game 2 by specifying details in the case of a tie, which may occur with positive probability because of bunching in the virtual valuation functions, and by mimicking the more complicated payment mechanism of the optimal auction. The winner is asked to volunteer two payments p_1 and p_2. Intuitively, the game must be such that the winner i has an incentive to offer $p_1 = t_i(b)$ and $p_2 = r_i(b)$, as the lowest unchallengeable prices. Besides this additional feature, the intuition behind Game 2' is the same as the one behind Game 2, presented in the previous section.

As should be expected, Game 2' admits several perfect Bayesian equilibria, however we have:[9]

Theorem 1 *: Game 2' (weakly) implements the optimal auction outcome.*

The strategies and beliefs that support the implementation equilibrium can be easily described. Bidder i bids up to his (monotonic) virtual valuation $H_i(v_i)$ when he is of type v_i; the winning bid therefore reveals the second highest virtual valuation. Then, the bidder volunteers payments $p_1 = t_i(b)$ and $p_2 = r_i(b)$ as required in the revenue-maximizing auction. He is disciplined in doing so by the

[9] This theorem is proved along with the characterization theorem below in the Appendix.

off-equilibrium-path threat of a challenge. When $p = p_1$ there is a possibility that there was a tie between winners; challengers' beliefs correspond to the truncation of $F_i(.)$ on $[t_i(b), \bar{v}_i]$, and a price $p = p_1 < t_i(b)$ is challenged by $q = t_i(b)$. When $p = p_2$ however, beliefs' support is $(r_i(b), \bar{v}_i]$ so that a price $p = p_2 < r_i(b)$ is challenged by $q = r_i(b)$.

As mentioned in the previous section, however, the previous theorem only asserts weak implementation of the optimal auction. We close this section with a complete characterization of the set of equilibrium outcomes of Game 2'; the description of these equilibria is in fact a generalization of the discussion in the end of the previous section.

Theorem 2 *: Each Perfect Bayesian equilibrium outcome of Game 2' is characterized by a profile of left-continuous non-decreasing functions $\mu_i(.)$ such that for all $b \geq 0$,*

$$\mu_i(b) \in \arg\max_{q \geq \mu_i(b)} [(q - b)[1 - F_i(q)]],$$

and of non-decreasing functions $\eta_i(.)$ defined by: $\eta_i(b) = \lim_{b' \downarrow b} \mu_i(b')$, such that:

(i) in the ascending auction, bidder i bids up to $b_i(v_i)$ defined as the value of b such that $v_i \in [\mu_i(b), \eta_i(b)]$, if it is not smaller than 0, or drops out immediately at 0 otherwise;

(ii) when he wins at winning bid b, bidder i submits prices $p_1 = \mu_i(b)$ and $p_2 = \eta_i(b)$;

(iii) there is no challenge.

For every such profile, there exists an equilibrium that yields the corresponding outcome.

This characterization theorem only provides a description of equilibrium paths but not a complete description of the strategies and beliefs involved. Many variations in the underlying strategies can correspond to equilibria so long as they generate, off-the-equilibrium path, the appropriate incentives not to deviate and as long as they do not imply a different behavior with a positive measure.[10]

Note that since $\mu_i(b') \in \arg\max_{q \geq \mu_i(b^0)}(q - b')[1 - F_i(q)]$ for b' converging toward b, we must also have: $\eta_i(b) \in \arg\max_{q \geq \eta_i(b)}(q - b)[1 - F_i(q)]$. The equilibrium underlying the results in Theorem 1 is therefore a limit case of the above equilibria with $\mu_i(.) = t_i(.)$. Moreover, for any equilibrium outcome, $\mu_i(b) \geq t_i(b)$ and consequently, $\eta_i(b) \geq r_i(b)$. The corresponding bidding functions $b_i(.)$ consequently satisfy: for all v_i, $b_i(v_i) \leq H_i(v_i)$. All equilibria of Game 2' involve a (weakly) less aggressive behavior from bidders in the ascending auction than in the limit equilibrium that implements the revenue-maximizing auction. We will rely on this property to refine the set of equilibria in the following section.

Note also that the pair of prices proposed by the winner of the first auction could a priori be used to signal the winner's type v_i. However, different valuations for the winner do not imply different (marginal) benefits from changing these prices and all equilibria will exhibit full pooling among all types that could have won the ascending auction; hence the continuation equilibria are simple enough and allow a complete characterization of the set of equilibrium outcomes.

[10]For example, if $\mu_i(b) = \eta_i(b) = \mu$ for $b \in (b', b")$ (therefore with $b' \geq 0$ and $b" \leq H_i(\mu)$), this corresponds to a jump in the bid function $b_i(.)$ which is not uniquely defined at μ. But the choice of any value for $b_i(\mu)$ in $(b', b")$ is innocuous for global equilibrium behavior.

5 Strict implementation and refinements

Game 2' admits a continuum of equilibria and may therefore induce a sub-optimal allocation as an equilibrium. This undermines our argument that, compared to sophisticated Maskin-type mechanisms, Game 2' is simple and applicable to real auction situations. This section addresses this issue of multiplicity.

Multiplicity of perfect Bayesian equilibria should not be a surprise. One can construct "unreasonable" equilibria by allowing potential challengers to have unreasonable out-of-equilibrium beliefs leading to high challenges q. In order to restrict beliefs out-of-the-equilibrium path, we can do two things: (i) we can slightly alter the structure of the game and engineer trembles in the extensive form of the game that nail down all beliefs; in other terms, we can achieve a unique equilibrium if we complexify the game in a way that is not practically meaningful, although, as will be seen, the complexified game could still be implemented in practice; or (ii) we can impose a refinement of the perfect Bayesian equilibrium concept that generates a unique equilibrium, i.e. we can argue that the equilibrium in Theorem 1 can be interpreted as the only reasonable equilibrium in Game 2'. We do both in the following, relying on an informal discussion and focusing on the regular case.

The important point we want to make is that the equilibrium that implements the revenue-maximizing auction is not any equilibria. As a limit equilibrium, it has a focal point property; moreover it is the unique equilibrium where beliefs on and off equilibrium are concentrated on $\{v_i$, such that $v_i \geq p\}$.

5.1 Adding trembles to the game

Consider first a slightly different version of Game 2'. With a very small probability ε, the stages presented in game 2 are replaced by what we shall call a trembling procedure, while with probability $1-\varepsilon$ the game follows its normal path, similar to Game 2'.

GAME 2' with ε-trembling procedure

1. Bidders participate in an ascending auction. With probability ε, the seller switches to the trembling procedure by randomly selecting a winner, irrespective of the bids submitted, while with probability $1-\varepsilon$ the game proceeds as in Game 2'. In both cases the winner i and the winning bid b are publicly disclosed, but bidders are not informed about whether the game is in its normal path or in the trembling procedure.

2. Along the normal path as in Game 2. Following the trembling procedure, the seller randomly sets a price p with $b < p$. If the winner i refuses to pay this arbitrary price p, the auction is called off. If w accepts to pay p, the seller publicly discloses p.

3. Along the normal path as in Game 2. Following the trembling procedure, other bidders are asked to challenge price p by quoting a higher price $q > p$. If no challenge is issued, i receives the unit at price p. If a challenge is issued at price q, i receives the unit at price p with a given probability β and, with probability $1-\beta$, he is asked to accept or reject price q. In this last situation, gains for the challenger are the same as in the original stage 3 of Game 2'.

The trembling procedure is constructed so that, at all possible price b at which i can win, no information set in stage 3 is unreached. So beliefs are fully determined by Bayes rule. Now as ε converges to 0, consider a corresponding sequence of equilibria in the perturbed game that converges to an equilibrium of the normal auction procedure characterized by $b_i(.)$. Consider, along the sequence, an information set at stage 3 which is inconsistent with the equilibrium strategy along the normal auction procedure: $p \neq b_i^{-1}(b)$. Observing this information set and for a small enough ε, challengers must then believe that the trembling procedure has been followed. Further, note that in the trembling procedure, the winner i is induced to get the good and pay p with positive probability, and can never get the good for a smaller price: it is a strictly dominant strategy for i to reject p if $v_i < p$. Moreover, if $v_i > p$, i can get the unit for price p and reject any challenge $q > p$; therefore, for $v_i > p$, it is a strictly dominant strategy for i to accept p. It follows that for all information sets inconsistent with the equilibrium strategy in the normal procedure, challengers must hold beliefs on v_i that coincide with the Bayesian updating of their prior conditional on $v_i \geq p$.

As we show now, this restriction on beliefs is sufficient to break all the equilibria characterized in Theorem 3, except the ones with $b_i^{-1}(b) = P_i(b)$ almost always. Suppose that an equilibrium is played with $b_i^{-1}(b) > P_i(b)$ for a set of b of positive measure [recall that we cannot have $b_i^{-1}(b) < P_i(b)$]. Let us consider the following deviation: when i wins at prices b such that $b_i^{-1}(b) > P_i(b)$, he submits $p = P_i(b)$, instead of $b_i^{-1}(b)$. Since $(p-b)[1-F_i(p)] > (q-b)[1-F_i(q)]$ for all $q > p$ whenever $p = P_i(b)$, challengers will never find profitable to challenge i if, realizing they face an information set off the equilibrium path, they hold beliefs concentrated on $\{v_i \geq P_i(b) = p\}$, as implied by the trembling procedure for small enough ε. Bidder i's deviation will then be strictly profitable: her price offer will not be challenged and she will get the unit at a strictly lower price, breaking the proposed equilibrium.

So if we let ε, the probability of switching to the trembling procedure, to be arbitrarily small, one can implement (uniquely) an equilibrium outcome which is arbitrarily close to the revenue-maximizing outcome.

5.2 Refining the set of equilibria

In this sub-section, we will restrict the set of Perfect Bayesian equilibria of Game 2' by imposing extra restrictions on beliefs off the equilibrium path. We apply here the concept of explicable equilibrium due to Reny [1992].

The idea underlying the notion of "explicable" equilibrium is that when a deviation is detected, the other participants must interpret this deviation not necessarily as an irrational move on the part of the deviator but, whenever possible, as the result of some confusion over which equilibrium is being played. Whenever possible, a deviation should be interpreted as a best-response to some other equilibrium. Formally, let B be some common standard of behavior and let π be some strategy profile in B. Now suppose that for $i \neq j$, (a) an information set, h, for j is inconsistent with i's strategy profile; (b) h is both consistent with i using the distinct pure strategies s and s'; (c) s is a best response relative to B while s' is not, where s is a best response to B if there exists an element $\gamma \in \text{co}B$ such that s is a best response against γ. Then, according to the notion of explicable equilibrium, if h is reached, j's reference about i's strategy should put zero probability on s' being played.

In the context of our game, the notion of explicable equilibrium has sufficient bite if we set B to be the set of Perfect Bayesian Equilibria. It can eliminate all equilibria characterized by a $b_i(.)$ function such that $b_i^{-1}(b) > P_i(b)$ with strictly positive measure. Consider such an equilibrium and suppose an information h is reached where i wins the auction at some price b and, as in the previous subsection, offer $p = P_i(b) < b_i^{-1}(b)$. We know that the strategy s used by bidder

i in the implementation equilibrium consists of bidding according to $H_i(.)$ and offering $p = P_i(b)$, and it is a best-response to the set of perfect Bayesian equilibria. However, any strategy s' for i that consists of bidding according to a $b_i(.)$ function such that $b_i^{-1}(b) > P_i(b)$ with positive probability, and of offering $p(b) = P_i(b)$ is not a best-response relative to the set of perfect Bayesian equilibria: $p(b) = P_i(b)$ lies off the equilibrium path and either this price is challenged, in which case i could have profitably proposed a higher p, or it is not challenged, in which case i did not followed an optimal bidding strategy since only types in $\left\{v_i \geq b_i^{-1}(b)\right\}$ bid up to b and may win at price b while it would have been profitable for $v_i \in \left(P_i(b), b_i^{-1}(b)\right)$ to win at price p. Hence, the challengers' inference about i's play must put zero probability on all strategies where i bids up to b only if $v_i \geq b_i^{-1}(b) > P_i(b)$, for any candidate $b_i(.)$-function. Since offering $p = P_i(b)$ is a strictly dominated strategy if $v_i < P_i(b)$, the challengers' inference must then be that $v_i \geq p = P_i(b)$. This again nails down beliefs off-the-equilibrium path. Following the discussion in the previous subsection, it selects a unique perfect Bayesian equilibrium outcome, the equilibrium that implements the optimal auction outcome.

6 Discussion

We have proposed a game form that implements the optimal auction in a relatively simple way without requiring extensive knowledge on the part of the auctioneer. The key features of Games 1, 2 and 2' are that the winner of the auction does not pay the winning price of the auction; she pays some price that is determined afterwards through some well-defined bargaining process. This is not very different from some current practices. Often the competitive process is meant only to identify a winner, the actual price and contract conditions are bargained afterwards between the interested parties.

The result of this paper has many limitations. We have restricted our attention to the case of private and independent values and the case of risk-neutral bidders. We also assume that one and only one unit is on sale, we do not consider how the logic here applies to multi-auctions with multi-unit demands. Our last concern relates to the repetition of these auctions. The presumption that participants are well-informed about the distributions of valuations of other participants reflects the notion that they all share a common experience and that these auctions are often repeated. If this is true, then collusion may arise: as a rule buyers may agree never to challenge each other. So, we view this paper as a first step in the pursuit of finding practical implementation procedures for optimal auctions.

References

[1] Bulow, J. and J. Roberts [1989]: “The simple Economics of Optimal Auction,” *Journal of Political Economy*; **97**; 1060-1090.

[2] Cramton, P. [1995]: ”Money out of Thin Air: The Nationwide Narrowband PCS Auction”, *Journal of Economics and Management Strategy,* **4**, 267-343.

[3] Donald, S.G. and H.J.Paarsch [1996]: “Piecewise pseudo-maximum likelihood estimation in empirical models of auction”, *International Economic Review*; **34**; pp.121-148.

[4] Fudenberg, D., D. Kreps and D. Levine [1988]: ”On the Robustness of Equilibrium Refinements”, *Journal of Economic Theory,* **44**, 354-380.

[5] Fudenberg, D. and J. Tirole [1991]: ”Perfect Bayesian Equilibria and Sequential Equilibria”, *Journal of Economic Theory,* **53**, 236-260.

[6] Glazer, J. and C.-A. Ma [1989]: ”Efficient Allocation of a “Prize”- King Solomon’s Dilemma”, *Games and Economic Behavior,* **1**, 222-233.

[7] Harsanyi, J. [1968]: ”Games with Incomplete Information Played by Bayesian Players”, I-II-III, *Management Sciences,* **14**, 159-182, 302-334, 486-502.

[8] Laffont, J.-J., H. Ossard and Q. Vuong [1995]: ”Econometrics of First-Price Auctions”, *Econometrica,* **63(4)**, 953-980.

[9] McAfee, P. and J. McMillan [1996]: ”Analyzing the Airwaves Auction”, *Journal of Economic Perspective,* **10(1)**, 159-176.

[10] McAfee, R.P., D.Quan and D. Vincent [1996]: ”How to set Minimum Acceptable Bids”, *mimeo.*

[11] Maskin, E. [1977]: "Nash Equilibrium and Welfare Optimality", mimeo.

[12] Mertens, J.-F. and S. Zamir [1985]:"Formulations of Bayesian Analysis for Games of Incomplete Information", *International Journal of Game Theory,* **14(1)**, 1-29.

[13] Moore, J. [1992]: "Implementation, Contracts and Renegotiation in Environments with Symmetric Information", in *Advances in Economic Theory: the 6th World Congress*, ed. J.J.Laffont, Cambridge University Press.

[14] Myerson, R. [1981], "Optimal Auction Design," *Mathematics of Operational Research*; **6**; 58-63.

[15] Palfrey, T. [1992]: "Implementation in Bayesian Equilibrium: the Multiple Equilibrium Problem in Mechanism Design", in *Advances in Economic Theory: the 6th World Congress*, ed. J.J.Laffont, Cambridge University Press.

[16] Reny, P. [1992], "Backward Induction, Normal Form Perfection and Explicable Equilibria," *Econometrica*; **60**(3); 627-650.

A Proof of Theorem 2

An equilibrium in Game 2' can be characterized by (i) bid functions $b_i(.)$ in the ascending auction, where $b_i(v_i)$ denotes the price at which bidder i with valuation v_i drops out; (ii) conditional on winning at the winning bid b and on his type v_i, price offers by i, $p_{i1}(b, v_i)$ and $p_{i2}(b, v_i)$; (iii) beliefs for potential challengers about i's valuation, conditional on (i, b, p) and on whether $p = p_1$ or $p = p_2$, and a decision rule which specifies given that information whether to challenge i or not and if so at which price $q^i(b, p, \{p = p_k\})$.

A.1 Sufficiency

We prove that for any profile of left-continuous increasing functions $\mu_i(.)$ and associated profile of functions $\eta_i(.)$ satisfying the condition in Theorem 5, one can construct strategies that sustain the corresponding outcome as an equilibrium.

The equilibrium strategies and beliefs are as follows. Bidder i bids up to b such that $v_i \in [\mu_i(b), \eta_i(b)]$. If i wins the initial auction at value b, she volunteers payments $p_1 = \mu_i(b)$ and $p_2 = \eta_i(b)$. These prices are unchallenged. However, if $p = p_1 < \mu_i(b)$, it is challenged and the challenger offers $q = \mu_i(b)$, and if $p = p_2 < \eta_i(b)$, the challenger offers $q = \eta_i(b)$. [11]

First, consider the challenger's beliefs when i wins at winning bid b and m bidders apart from i dropped out at b. If $p = p_1$, it must be that either i also dropped out at b and was randomly selected with probability $\frac{1}{m+1}$ or that i dropped out above b but $p = p_1$ was announced which had probability $\frac{1}{m+1}$. The posterior be-

[11] We do not specify here what happens if $p = p_1 > \mu_i(b)$ or $p = p_2 > \eta_i(b)$ since it $>$ is irrevelant to the strategic analysis below. i will never offer a price above the lowest unchallenged price.

liefs therefore coincide with the Bayesian updating of prior beliefs conditional on $\{v_i \geq \mu_i(b)\}$. If $p = p_2$ is announced, challengers must understand that i dropped above b, and posterior beliefs must be the Bayesian updating of prior beliefs conditional on $\{v_i > \eta_i(b)\}$.

Given these posterior beliefs, whenever $p = p_1$, it is sequentially rational to challenge p if $p < \mu_i(b)$ and offer $q = \mu_i(b)$ since $\mu_i(b) \in \arg\max_{q \geq \mu_i(b)}(q - b)[1 - F_i(q)]$. Similarly, whenever $p = p_2$, it is optimal to challenge p if $p < \eta_i(b)$ and offer $q = \eta_i(b)$ since $\eta_i(b) \in \arg\max_{q \geq \eta_i(b)}(q - b)[1 - F_i(q)]$.

Given the response of potential challengers, if i wins the initial auction at price b, her best strategy is to offer $p_1 = \mu_i(b)$ and $p_2 = \eta_i(b)$ unless $v_i < \mu_i(b) - \Delta$ in which case she offers a challengeable price, pays Δ and ends up not receiving the good. So let $\hat{b}$ be the value at which the last of all other participants drops out. Bidder i gets negative payoffs if he wins and $v_i < \mu_i(\hat{b})$ and he gets positive payoffs if he wins and $v_i > \eta_i(\hat{b})$ or if he drops out exactly at $\hat{b}$ and $\mu_i(\hat{b}) < v < \eta_i(\hat{b})$. His best-response is indeed to bid up to the value b such that $v_i \in [\mu_i(b), \eta_i(b)]$.

A.2 Necessity

We proceed through a series of claims.

First, we define for all i, $Y_i = \{x | \exists v \text{ such that } b_i(v) = x\}$. Y_i denotes the support of all bids that i may issue. We also define $Y_{-i} = \cup_{j \neq i} Y_j$. So if $b \in \text{int}Y_{-i}$, then there exists strictly positive probability that some bidder other than i bids in an set with non-empty interior which closure includes b. Finally, let $dG_i(b)$ denotes the equilibrium measure corresponding to the probability that i wins at the winning bid b, for some equilibrium. Note that $dG_i(.)$ may not be absolutely continuous w.r.t. the Lebesgue measure since it may include Dirac mass points in case of ties;

in other words, $G_i(.)$ may exhibit a countable number of discontinuities, at which it may even be right- and left-discontinuous.

Claim (i): *For dG_i-almost all $b \in Y_{-i}$, define the following:* $\mu_i(b) \equiv \sup\{v_i, b_i(v_i) < b\}$ *and* $\eta_i(b) \equiv \sup\{v_i, b_i(v_i) \leq b\}$; *then, in equilibrium* $b_i(v_i) < b$ *if* $v_i < \mu_i(b)$, $b_i(v_i) = b$ *if* $\mu_i(b) < v_i < \eta_i(b)$, *and* $b_i(v_i) > b$ *if* $v_i > \eta_i(b)$.

The claim asserts that in equilibrium, bid functions must necessarily be increasing when they are relevant. It implies that if v_i and v_i' are such that $v_i < v_i'$, then $b_i(v_i) \leq b_i(v_i')$ except possibly if $(b_i(v_i'), b_i(v_i)) \cap Y_{-i} = \emptyset$. Monotonic bid function implies that beliefs on the equilibrium path must also be monotonic. This monotonicity property is reminiscent of standard results in more conventional auction settings.

Proof of Claim (i).

Let $U_i(v_i, b_i)$ denote the expected payoffs of bidder i of type v_i who bids up to b_i. It depends on the highest bid among all other participants and the realization of the final price he will be charged if he wins, which may be random. Consider two possible bids b_i and b_i', with $b_i > b_i'$ and $b_i \in \text{int}Y_{-i}$. If $U_i(v_i, b_i) \geq U_i(v_i, b_i')$, we can write:

$$0 \leq \int_{b_i'}^{b_i} u_i\left(v_i, p_1(b, v_i), p_2(b, v_i), b\right) dG_i(b).$$

where the index $u_i(v_i, p_1, p_2, b)$ stands for bidder i's expected payoff when he is of type v_i, wins at winning bid b and proposes prices (p_1, p_2); it depends upon the strategies and beliefs used in the continuation game by the other bidders after i has won at winning bid b. This index $u_i(v_i, p_1, p_2, b)$ may coincide with one of i's two equilibrium proposals, $u_i(v_i, p_1, p_2, b) = v_i - p_1$ or $v_i - p_2$, or with an accepted challenge, $u_i(v_i, p_1, p_2, b) = v_i - q^i(b, p, \{p = p_k\}) - \Delta$, or else with a rejected challenge, $u_i(v_i, p_1, p_2, b) = -\Delta$.

$u_i(v_i, p_1, p_2, b)$ is non-decreasing in v_i and even strictly increasing in v_i when it is non-negative. The interval (b_i', b_i) has strictly positive dG_i-measure since $b_i \in \text{int}Y_{-i}$, and then the inequality implies that within (b_i', b_i), the integrand is non-negative on a set of positive dG_i-measure on which it is then strictly increasing in v_i. Therefore, for all $v_i' > v_i$, if $U_i(v_i, b_i) \geq U_i(v_i, b_i')$,

$$0 < \int_{b_i'}^{b_i} u_i\left(v_i', p_1(b, v_i), p_2(b, v_i), b\right) dG_i(b)$$

and then $U_i(v_i', b_i) > U_i(v_i', b_i')$. The result follows.

Note also that if b_i is a mass point of dG_i, a similar argument holds. ■

Claim (ii): *Suppose that i wins at winning bid $b \in Y_{-i}$, then along the equilibrium path, i offers $p_1(b, v_i) = \mu_i(b)$ and $p_2(b, v_i) = \eta_i(b)$ which are almost never challenged, for dG_i-almost all $b \in Y_{-i}$.*

Proof of Claim (ii).

For $b \in \text{int}Y_{-i}$, let $k_1(b, v_i, i)$ (resp. $k_2(b, v_i, i)$) denote the effective price (including the fee Δ if challenged, equal to $v_i + \Delta$ if the challenge is rejected) paid by i along the equilibrium path whenever he wins the auction at price b and $p = p_1$ (resp. $p = p_2$). These prices may depend upon v_i since price proposals may themselves depend upon v_i. We will concentrate on $k_1(b, v_i, i)$ in the proof, but the proof is similar for $k_2(b, v_i, i)$.

Let $k_1^*(b, i) \equiv \inf_{\{v_i \geq \mu_i(b)\}} k_1(b, v_i, i)$. We show that $k_1^*(b, i) = \mu_i(b)$ almost always, in the sense of dG_i. Suppose that $k_1^*(b, i) < \mu_i(b)$, i.e. there exists one $v_i \geq \mu_i(b)$ for which the effective price along the equilibrium path is less than $\mu_i(b)$. Clearly, it cannot correspond to a rejected challenge in which case $k_1(b, v_i, i) = v_i + \Delta > \mu_i(b)$. It cannot correspond to an accepted challenge or to an unchallenged price either, since the challenger would strictly benefit from challenging at a price q such that $k_1(b, v_i, i) < q < \mu_i(b)$, which would surely be accepted given his posterior beliefs concentrated on $\{v_i \geq \mu_i(b)\}$.

Suppose now that there exists an interval $(b_0, b_0 + \epsilon) \subset Y_{-i}$ and $\delta > 0$ such that:

$$\forall v_i \in (\mu_i(b_0), \mu_i(b_0) + \delta), \forall b \in [b_0, b_0 + \epsilon), \quad k_1^*(b, i) > v_i.$$

This in turn implies that a bidder i of type v_i in this right-neighborhood of $\mu_i(b_0)$ would have been strictly better off by submitting a bid strictly lower than b_0, which contradicts the definition of $\mu_i(.)$. It follows that, in equilibrium, $k_1^*(b, i) = \mu_i(b)$ for almost all $b \in \text{int}Y_{-i}$ after history (i, b).

From this, it follows that if there is a challenge in equilibrium, it must be accepted and must occur at price $q = \mu_i(b) - \Delta$. But this price cannot correspond to a rational challenge since a challenger knows that the winner i must have valuation $v_i \geq \mu_i(b)$ and would therefore accept a slightly higher challenge price with probability 1. To conclude, in equilibrium, it is necessary that i winning at b volunteers $p_1 = \mu_i(b)$ (and $p_2 = \eta_i(b)$) and that there is no challenge whatever her type $v_i \geq \mu_i(b)$ $(v_i \geq \eta_i(b))$.

Note finally that the assumption that $b \in \text{int}Y_{-i}$ was only used to exhibit a positive dG_i-measure set of bids on which $k_1(b, v, i) > v$. If b instead is a mass point of Y_{-i} not in $\text{int}Y_{-i}$, a similar argument leads also to $k_1(b, v, i) = \mu_i(b)$ almost always (in the sense of dG_i). ■

Claim (iii): *Along the equilibrium path, when i wins the auction at b for dG_i-almost every $b \in Y_{-i}$, then $\mu_i(b)$ must be such that:*

$$\mu_i(b) \in \arg \max_{q \geq \mu_i(b)} (q - b)[1 - F_i(q)] \tag{3}$$

Proof of Claim (iii).

Given Claim (i), when a challenger faces the winner i at winning bid b and with $p = p_1$, he should have beliefs corresponding to the updating of prior beliefs conditional on $\{v_i \geq \mu_i(b)\}$ and should not find any profitable challenge. If there were a $q > \mu_i(b)$ such that $(q-b)[1-F_i(q)] > (\mu_i(b)-b)[1-F_i(\mu_i(b))]$, there would exist such a strictly profitable challenge against p_1. The same holds for $\eta_i(b)$. Hence the Claim. ■

This completes the proof of the theorem. ■