The Coalitional Nash Bargaining Solution[*]

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Abstract

The coalitional Nash bargaining solution is defined to be the core allocation for which the Nash product is maximal. We consider a non-cooperative model in which one team may form. The grand team, consisting of all players, generates the largest surplus. But a smaller team may form. We show that as players get more patient if an efficient and stationary equilibrium exists, it must deliver payoffs that correspond to the coalitional Nash bargaining solution. We also characterize when an efficient and stationary equilibrium exists. We find that existence requires conditions that go beyond the non-emptyness of the core.

1 Introduction

We study a non-cooperative model of bargaining in which one team may form. The team that forms will generate a surplus or revenue. A team forms when it agrees on how to share the surplus or revenue it generates. The grand team consisting of all players is the one that generates the largest surplus. However, a team (or coalition) smaller than the grand team may form and players left outside the winning team get 0. Players bargain under the threat that a team smaller than the grand team may form.

We model bargaining by assuming that in each period, every player has an equal chance of being selected to make an offer (about a team and sharing device). If the offer is accepted by all tentative team members, it is implemented and bargaining stops. Otherwise, one

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proceeds to the next stage, which has the same structure. Every player discounts future payoffs according to the same discount factor assumed to be close to 1.

We ask ourselves: 1) Under what conditions does there exist a stationary equilibrium that is almost efficient in the sense that the grand team is almost always formed instantaneously? 2) When an efficient equilibrium exists, what are the resulting payoffs obtained by the various players?

We obtain the following characterizations. In an efficient stationary equilibrium, the profile of payoffs lies in the core,[1] and the product of players' payoffs (or Nash product) is maximal among all core allocations. Moreover, for an efficient stationary equilibrium to exist, it should be that 1) the core is non-empty and 2) as one increases uniformly (i.e. by the same (small) amount Δ) the surplus generated by every team S, the maximal Nash product obtained over core allocations increases as well.

The first condition is familiar.[2] The second condition however has no counter part in the literature. We illustrate through examples that this second condition may fail even when the core has a non-empty interior.

We shall refer to the allocation that maximizes the Nash product among all core allocations as the *coalitional Nash bargaining solution.* It can be viewed as a multi-player extension of the Nash bargaining solution. As in Binmore-Rubinstein-Wolinsky (1986), the connection between the non-cooperative bargaining game and the Nash maximization program is established by comparing the equilibrium conditions of the bargaining game[3] to the first order conditions characterizing the solution of the maximization program.

Our solution differs from the Shapley value - the most popular extension of the Nash bargaining solution to the multi-player case - in several respects. First, the coalitional Nash bargaining solution is a selection of the core whereas there is no guarantee in general that the Shapley value lies in the core. Second, in the coalitional Nash bargaining solution there can be at most as many coalitions as there are players that matter for the determination of the outcome.[4] This is in sharp contrast with the Shapley value in which the value of all

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[1] That is, it is such that every coalition S receives at least the revenue $v(S)$ generated by S and the grand coalition N receives no more than the feasible revenue $v(N)$.

[2] In an efficient equilibrium, any player must be making offers to the grand team with a probability close to 1. If a team S could generate a surplus larger than the sum of equilibrium payoffs accrued to that team, any member of that team S would be strictly better off making an offer to S rather than to the grand team. It follows that no coalition S can be blocking in an asymptotically efficient equilibrium.

[3] That is, the equilibrium conditions derived from looking at one shot deviations.

[4] Call x_i the payoff obtained by i and let n be the total number of players. A coalition S affects the coalitional Nash bargaining solution if $\sum_{i\in S} x_i = v(S)$ where $v(S)$ is the surplus generated by S. There can

coalitions (there are $2^n - 1$ such coalitions where n is the total number of players) matter for the determination of the outcome.

From a non-cooperative game-theoretic viewpoint, the finding that only a subset of coalitions matter should be related to the idea that not all teams (or coalitions) can *credibly* form in equilibrium. Those coalitions that pin down the coalitional Nash bargaining solution are those teams that can credibly form in equilibrium. From this perspective, our analysis can be viewed as generalizing the insight due to Binmore-Shaked-Sutton (1989) about the role of outside options in two-person bargaining: Binmore-Shaked-Sutton (1989) pointed out that outside options should have no effect on the equilibrium outcome if they are not credible (because they deliver payoffs that are inferior to the equilibrium payoffs obtained when outside options are absent). In our analysis too, not all teams can credibly form and our characterization allows us to understand simply which teams can credibly form in equilibrium (these are the coalitions for which the constraint that they should not be blocking is binding for the determination of the core allocation that maximizes the Nash product).

In the rest of the paper, we present in more detail the model, the coalitional Nash bargaining solution and our main results. We also apply our analysis to convex games and to seller-buyers games which are buyer- submodular, and discuss the related literature.

2 The Model

We consider n players labelled $i = 1, .., n$. Any subset S of these players may form a team. The surplus that such a team generates is denoted $v(S)$. Once a team forms the game stops and no further team can form. We let 0 be the payoff that player i obtains as long as no team has formed yet where 0 is also the payoff player i gets when he is not part of the winning team. We denote by $N = \{1, ..., n\}$ the set that comprises all players, and by $\mathcal{S}$ the set of strict subsets of N.

The bargaining game. In any period $t = 1, ...,$ a proposer is selected randomly. Draws are independent, and each player i has an equal chance $1/n$ of being selected. A proposer chooses a subset S of agents, possibly equal to $N = \{1, .., n\}$. He makes a proposal $x^S = (x_i^S)_{i \in S}$ to share the surplus $v(S)$ that satisfies the feasibility condition:

$$\sum_i x_i^S \leq v(S)$$

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be no more than n such identities for generic value of $v(S)$, since there are only n unkowns.

Players in S are asked whether they accept or reject the proposal. If they all accept, the team S forms and each team member $i \in S$ gets a payoff equal to x_i^S (non-team members get 0). If one or more players in S rejects the proposal, the game moves to the next period, which has the same structure. All players discount future payoffs with the same discount factor $\delta < 1$. So in case a coalition S forms at date t and accepts x^S, ex ante payoffs are $\delta^{t-1} x_i^S$ if i belongs to the coalition S, or 0 if player i does not belong to S.

Throughout the paper, we assume that the grand team is the only efficient coalition. That is,

$$v(N) > v(S) \text{ for all } S \neq N.$$

We call $\mathcal{P}$ the set of possible proposals, and $\mathcal{A}$ the set of possible acceptance rules. In principle, a strategy for player i specifies, at each date t, and for every history of the game up to date t, a proposal $(S, x^S) \in \mathcal{P}$ (possibly a mixed proposal) in case i is the proposer, and an acceptance rule in $\mathcal{A}$ in case i is not a proposer. We will however restrict our attention to *stationary* equilibria of this game, where each player adopts the same proposal and acceptance rule at all dates. We will analyze when such equilibria can be almost efficient as δ tends to 1, and we will characterize the payoffs obtained by the various players in the limit as δ tends to 1 of such equilibria.

We first introduce a few preliminary definitions. We define an allocation as a vector of payments $x = (x_1, ..., x_n)$, and we let $x(S) = \sum_{i \in S} x_i$. Given our assumption that $v(N) = \max_S v(S)$, an allocation x is *efficient* if

$$x(N) = v(N).$$

We turn now to the notion of core stability. An allocation x is *feasible* if

$$x(N) \leq v(N).$$

It cannot be blocked by a coalition S if

$$x(S) \geq v(S)$$

Definition 1. The *core* consists of allocations x that are feasible and that cannot be blocked by any coalition $S \in \mathcal{S}$.

We denote by $\mathcal{C}$ the set of core allocations when it is non-empty. And we further define:

Definition 2. Assume the core is non-empty. The *coalitional Nash bargaining solution* is the core allocation x^* that maximizes the Nash product.[5] That is:

$$\{x^*\} = \arg\max_{x \in \mathcal{C}} \prod_{i \in N} x_i.$$

Definition 3. The bargaining game has an *asymptotically efficient equilibrium* with limit value u if there exists a sequence of pairs of discount factor and associated equilibrium value vector $(\delta^{(k)}, u^{(k)})$ with the property that $\delta^{(k)}$ tends to 1, $u^{(k)}$ tends to u, and $u(N) = v(N)$.

3 Main results

A first observation is that when the core is empty, there is no way one can sustain a stationary equilibrium that is approximately efficient as δ tends to 1.

Proposition 0. *When the core is empty there cannot exist an asymptotically efficient equilibrium.*

This is a fairly simple observation. Assume by contradiction that there exists an asymptotically efficient equilibrium with limit value u. When a coalition other than N is formed, there is an efficiency loss at least equal to $\min\limits_{S \neq N} (v(N) - v(S)) > 0$, so coalitions other than N should almost never form. This implies that when a player, say i, is selected to make a proposal, he must be getting a continuation equilibrium payoff arbitrarily close to $v(N) - \sum\limits_{j \neq i} u_j = u_i$.

Now observe that since the core is empty and since $u(N) = v(N)$, there must exist a coalition S^* such that $v(S^*) > u(S^*)$. Thus, any player i in S^* when selected to make an offer can get a payoff arbitrarily close to $v(S^*) - u(S^* \setminus \{i\}) = u_i + v(S^*) - u(S^*)$ by approaching coalition S^*. Since $u_i + v(S^*) - u(S^*)$ is bounded away from u_i, it follows that $i \in S^*$ would not propose coalition N, thereby yielding a contradiction.

In line with the previous literature on coalition formation, the above claim asserts that the existence of an asymptotically efficient equilibrium requires the non-emptyness of the core. Note that the argument also implies that an asymptotically efficient equilibrium must have a limit value in the core. From now on we will assume that the core has a non-empty interior.

A1: The core has a non-empty interior.

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[5] It is uniquely defined because the core is a convex set.

Our main findings are two fold. First, we show that *A1* alone is not sufficient to guarantee the existence of an asymptotically efficient equilibrium. Second, when an asymptotically efficient equilibrium exists, we show that it must coincide with the coalitional Nash bargaining solution introduced in Section 2.

To state our results, it is convenient to define the Δ-core as the set of allocations x that satisfy the "relaxed" constraints:

$$\begin{aligned} x(N) &\leq v(N) - \Delta \\ x(S) &\geq v(S) - \Delta \text{ for all } S \in \mathcal{S} \end{aligned}$$

We denote it $\mathcal{C}(\Delta)$. In what follows, Δ will be a non negative scalar. When we shall move to the equilibrium analysis of our coalition formation game, we will interpret Δ as the inefficiency that arises in equilibrium.

Let $\mathcal{N}(\Delta)$ denote the maximum of the Nash product among allocations in the $\Delta-$core, that is:

$$\mathcal{N}(\Delta) = \max_{x \in \mathcal{C}(\Delta)} \prod_{i \in N} x_i.$$

Increasing Δ from 0 has two effects. It reduces total welfare (measured as the sum of the utilities). This would seem to decrease the Nash product. But it also relaxes the constraints imposed by the coalitions. The resulting effect on the Nash product is thus ambiguous in general.

Whether equilibria can be efficient when players get very patient precisely depends on whether increasing Δ reduces the maximum Nash product $\mathcal{N}(\Delta)$ or not.

Formally, we define the following property:

> *P1: There exists Δ_0 such that for all $\Delta \in (0, \Delta_0)$, $\mathcal{N}(\Delta)$ is strictly decreasing in Δ.*

Our main results are stated in the following two propositions.

Proposition 1. *Let A1 hold. If P1 holds, then there exists an asymptotically efficient equilibrium. If P1 does not hold then all stationary equilibria of the bargaining game remain bounded away from efficiency.*

Proposition 2. *Let A1 hold. Assume there exists an asymptotically efficient equilibrium with limit value u. Then (P1 must hold and) u must coincide with the coalitional Nash bargaining solution.*

Whether property P1 holds or not depends on the characteristic function $v(\cdot)$. Example 1 provides a class of games where P1 holds. Example 2 illustrates a case in which P1 does *not* hold. According to Proposition 1, inefficiencies must then arise in any stationary equilibrium in this case. We will come back after the presentation of the proof of our main results why inefficiencies must arise in Example 2.

Example 1. Assume that there is a key player i^*, that is, a player such that $v(S) = 0$ if $i^* \notin S$. Then condition P1 holds.[6]

Example 2. Let $n = 5$. Assume $v(N) = 1$ and $v(ik) = w$ when $i \in \{1, 2\}$ and $k \in \{3, 4, 5\}$. Other coalitions yield 0. Let us compute $\mathcal{N}(\Delta)$. At the optimum, it must be that $x_1 = x_2 = x$ and $x_3 = x_4 = x_5 = y$.[7] Hence we have:

$$\mathcal{N}(\Delta) = \max_{\substack{x,y \\ 2x+3y\leq 1-\Delta \\ x+y\geq w-\Delta}} x^2 y^3$$

Choose $w < 1/2$ so that $\mathcal{C}$ has a non-empty interior. When the second constraint does not bind, $\mathcal{N}(\Delta) = (\frac{1-\Delta}{5})^5$, which is decreasing in Δ. When both constraints bind,

$$\mathcal{N}(\Delta) = (3w - 1 - 2\Delta)^2(1 - 2w + \Delta)^3$$

Taking the derivative of the RHS with respect to Δ at $\Delta = 0$, we get that

$$\begin{aligned} \left.\frac{d\mathcal{N}(\Delta)}{d\Delta}\right|_{\Delta=0} &> 0 \text{ when } w \in (\frac{7}{17}, \frac{1}{2}), \text{ and} \\ \left.\frac{d\mathcal{N}(\Delta)}{d\Delta}\right|_{\Delta=0} &< 0 \text{ when } w < \frac{7}{17}. \end{aligned}$$

So Property $P1$ holds on $(0, \frac{7}{17})$, but it does not hold on $(\frac{7}{17}, \frac{1}{2})$. Intuitively, when w gets close to 1/2, core allocations give a payoff to players 3,4,5 that approaches 0. By increasing Δ, the total surplus to be shared is decreased, but relaxing the core constraint $x(ij) \geq w$ permits to increase the payoff to players 3, 4, 5, hence to increase the Nash product.

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[6] Take the coalitional Nash bargaining solution x for some Δ_0 and consider $\Delta < \Delta_0$ (which corresponds to assuming that the surplus of all teams is increased by $\Delta_0 - \Delta$). The vector y defined as $y_{i^*} = x_{i^*} + \Delta_0 - \Delta$ and $y_i = x_i$ for all $i \neq i^*$ is an element of the Δ-core. Since $\prod_{j\in N} y_j > \prod_{j\in N} x_j$, we conclude that $\mathcal{N}(\Delta) > \mathcal{N}(\Delta_0)$, hence that P1 holds.

[7] For example, it cannot be that $x_1 > x_2$ at the optimum, otherwise $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, x_4, x_5)$ would also be in $\mathcal{C}(\Delta)$, but yielding a larger Nash product.

4 The Coalitional Nash Bargaining solution.

We explore the properties of the coalitional Nash bargaining solution through a characterization and examples, and relate it to other familiar solutions.

Another characterization. Consider a vector of non-negative weights $\mu = (\mu_S)_{S \in \mathcal{S}}$. We denote by $\mathcal{S}_i$ (respectively $\widetilde{\mathcal{S}}_i$) the set of coalitions in $\mathcal{S}$ to which i belongs (respectively *does not* belong), and we let

$$m_i^\mu = \sum_{S \in \widetilde{\mathcal{S}}_i} \mu_S.$$

The following Proposition provides a characterization of the Coalitional Nash bargaining solution, which is provided by considering the first-order conditions of the Lagrange maximization associated with the maximization of the Nash product.

> **Proposition 3.** *Let* $A1$ *and* $P1$ *hold. Then there exists a set of coalitions* $\mathcal{S}^* \subset \mathcal{S} \setminus N$*, an allocation* x^*, a vector of weights μ^* *and a scalar* $\alpha \in \{0,1\}$ *with the properties that*
>
> $$\begin{aligned} x^* &\in \mathcal{C} && (1)\\ \forall S &\in \mathcal{S}^*,\ \mu_S > 0 \text{ and } x^*(S) = v(S) && (2)\\ \forall S &\notin \mathcal{S}^*,\ \mu_S = 0 && (3)\\ x^*(N) &= v(N) && (4)\\ (\alpha + m_i^\mu) x_i^* &= (\alpha + m_1^\mu) x_1^* \text{ for all } i. && (5) \end{aligned}$$
>
> *Besides, the vector* x^* *is uniquely defined and coincides with the coalitional Nash bargaining solution.*

From the viewpoint of our non-cooperative game (to be analyzed later on), the set $\mathcal{S}^*$ will be interpreted as the set of credible coalitions, and μ_S will be interpreted as the strength of coalition S. Specifically, a coalition S is *credible* if it has a strictly positive strength $\mu_S > 0$. In the next Section, we will show that in the bargaining game, the set of credible coalitions coincides with the coalitions that are proposed with positive probability in equilibrium.[8] Also the strength of a coalition will be related to the probability that it is proposed in equilibrium. Note that, by condition (2), if S is a credible then we must have $x^*(S) = v(S)$

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[8] That probability will tend to 0 as players get very patient. So the fact that coalitions other than N are proposed will not generate inefficiencies at the limit.

We make three observations about credible coalitions:

(i) For generic values of $v(S)$, there are at most $N-1$ coalitions $S \neq N$ that may be credible. Indeed, we have noted that for each credible coalition S, one must have that $\sum_{i \in S} x_i = v(S)$. Given that $x(N) = v(N)$ holds and that there are only n unknowns $x_1,...x_n$, there can be at most $n-1$ coalitions $S \neq N$ that can credibly form.

(ii) For generic values of $v(S)$, if a coalition S is not credible then $x^*(S) > v(S)$, and reducing slightly the value of $v(S)$ does not affect the solution, as the same (x^*, μ^*) solve the constraints and the solution is unique.

(iii) Call $\mathcal{S}_i^*$ the set of credible coalitions to which i belongs. If $\mathcal{S}_i^* \supset \mathcal{S}_j^*$, then from (5) and the expression of m_i^μ, we get: $x_i^* \geq x_j^*$. So in particular, if $\mathcal{S}_i^* = \mathcal{S}_j^*$, then $x_i^* = x_j^*$.

These observations allow us to differentiate the coalitional Nash bargaining solution from solutions like the Shapley value or any solution in which players' payoffs are affected by marginal perturbations of the value of any coalition. More precisely, the Shapley value is defined as an allocation x^{sh} satisfying:[9]

$$x_i^{sh} = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right] \tag{6}$$

From (6), it is readily verified that the value $v(S)$ of all coalitions S affect x_i^{sh} in the sense that x_i^{sh} is modified by a slight perturbation of any $v(S)$. By contrast, from observation (ii) we know that a slight perturbation of $v(S)$ when S is not credible has no effect on the coalitional Nash bargaining solution and from observation (i) we know that there are at most $n-1$ credible coalitions other than N (out of a total of $2^n - 1$ coalitions). It follows that the Shapley value and the coalitional Nash bargaining solution cannot coincide in general.

The idea that not all coalitions matter for the determination of a bargaining equilibrium already appears in the literature on two-person bargaining with outside options (Binmore-Rubinstein-Wolinsky (1989)) in which the outside option of a player can be interpreted as the value of the coalition composed of this player only. We explain below how the above program specializes within our framework when there are only two players.

The two player case. Let $n = 2$, $v(N) = 1$ and $v(1) = w_1$, $v(2) = w_2$ with $w_1 + w_2 < 1$, and $w_1 \geq w_2$. There are two cases:

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[9] Observe that the Shapley value is defined whether or not the core is empty. By contrast, the coalitional Nash bargaining solution can only be defined if the core is non-empty.

- $\mathcal{S}^* = \emptyset$. Then $\mu_S = 0$ for all $S \neq N$, hence

$$x_1^* = x_2^* = \frac{1}{2}.$$

  This is the solution so long as $w_2 \leq w_1 \leq \frac{1}{2}$.

- $\mathcal{S}^* = \{1\}$. Then $\mathcal{S}_1^* = \mathcal{S}_2^*$ so

$$x_1^* = w_1 \text{ and } x_2^* = 1 - w_1.$$

  Since $\mu_1 \geq 0$ and $\mu_2 = 0$, (5) requires $x_1^* \geq x_2^*$, hence

$$w_1 \geq 1/2$$

As in the work on two-person bargaining with outside option, we find that as long as $w_i \leq 1/2$, it is not credible to take the outside option, and the outcome remains equal to $(1/2, 1/2)$. Once a player may get $w_i \geq 1/2$ he must be getting at least that amount, however, in equilibrium he cannot get more than that amount, because otherwise, the outside option would cease to be credible.

In a sense, our analysis suggests how to extend Binmore et al.'s analysis to the case of more than two players and it gives a recipe to find out which coalitions are binding as a function of the values of the various coalitions (that is, the binding coalitions are those that are associated with a strictly positive Lagrange multiplier in the Nash product maximization program). To illustrate the idea further, consider now the following three-player scenario.

A *three player example.* Let $n = 3$, $v(N) = 1$, $v(12) = w_2$, $v(13) = w_3$. We assume that $w_3 \leq w_2 < 1$, and that all other coalitions yield 0. In that example, player 1 is a key player: he must be part of any coalition that generates a positive surplus. Which coalitions are credible depends on the values of w_2 and w_3. We characterize below the coalitional Nash bargaining solution and the corresponding credible coalitions as a function of (w_2, w_3) by looking at all possible sets of credible coalitions and seeing what constraints on (w_2, w_3) apply.

- $\mathcal{S}^* = \emptyset$. Then $\mu_S = 0$ for all $S \neq N$, hence

$$x_1^* = x_2^* = x_3^* = \frac{1}{3}.$$

  This is the solution so long as $w_3 \leq w_2 \leq \frac{2}{3}$.

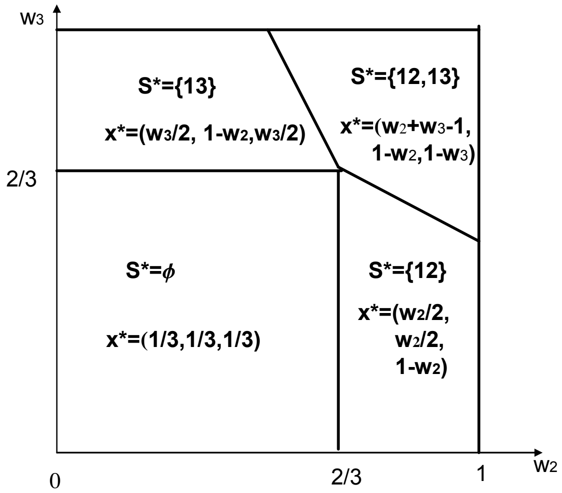


Figure 1:

- $\mathcal{S}^* = \{12\}$. Then $\mathcal{S}_1^* = \mathcal{S}_2^*$ so $x_1^* = x_2^*$. And $\mu_{12} > 0$, so $x_1^* + x_2^* = w_2$. It follows that

$$x_1^* = x_2^* = \frac{w_2}{2} \text{ and } x_3^* = 1 - w_2$$

  This is the solution as long as $x^*(13) \geq w_3$, and $x_1^* \geq x_3^*$, that is,

$$1 - \frac{w_2}{2} \geq w_3 \text{ and } w_2 \geq \frac{2}{3}$$

- $\mathcal{S}^* = \{13\}$. This requires (see above, exchanging the role of 2 and 3) $1 - \frac{w_3}{2} \geq w_2$ and $w_3 \geq \frac{2}{3}$. Since $w_2 \geq w_3$, these inequalities are not compatible.

- $\mathcal{S}^* = \{12, 13\}$. Then $x_1^* + x_2^* = w_2$ and $x_1^* + x_3^* = w_3$ so

$$x_1^* = w_2 + w_3 - 1,\ x_2^* = 1 - w_3,\ x_3^* = 1 - w_2,$$

  and (5) requires $x_1^* \geq x_2^*$ and $x_1^* \geq x_3^*$, that is:

$$w_3 + \frac{w_2}{2} \geq 1.$$

These cases are summarized in Figure 1.

To sum up, so long as both coalitions 12 and 13 yield less than 2/3, they are not credible and thus w_2 and w_3 cannot affect the outcome of the bargaining game. When at least one coalition, say 12, may get at least 2/3, that coalition becomes credible. However in equilibrium, players 1 and 2 cannot jointly obtain more than w_2 (because if it were the case, the coalition would cease to be credible). Observe that it is not necessary that a coalition gets at least 2/3 to be credible. If $v(12)$ is large, 3 must be getting a small payoff, making the coalition 13 attractive to player 1. (See Figure 1).

Observe that in the three-player example analyzed above, the solution is solely determined by the set $\mathcal{S}^*$of credible coalitions and not by the strength μ_S of the credible coalitions $S \in \mathcal{S}^*$. This need no longer be the case when we move to a 4-player bargaining situation: then condition (5) (that involves the vector of strength μ) will sometimes be necessary to compute the solution. This is illustrated below.

A 4-player example. We provide an example where, unlike the previous 2- and 3-player examples, finding the solution requires computing the coalition strength vector.

Let $v(N) = 1$, $v(12) = w_{12}$, $v(13) = w_{13}$. Other coalitions yield 0. We consider the case where both 12 and 13 would be credible coalitions. This occurs if and only if $w_{12} > 1/2$ and $w_{13} > 1/2$. The solution must satisfy:

$$\begin{aligned} 1 &= x_1^* + x_2^* + x_3^* + x_4^* \\ w_{12} &= x_1^* + x_2^* \\ w_{13} &= x_1^* + x_3^*. \end{aligned}$$

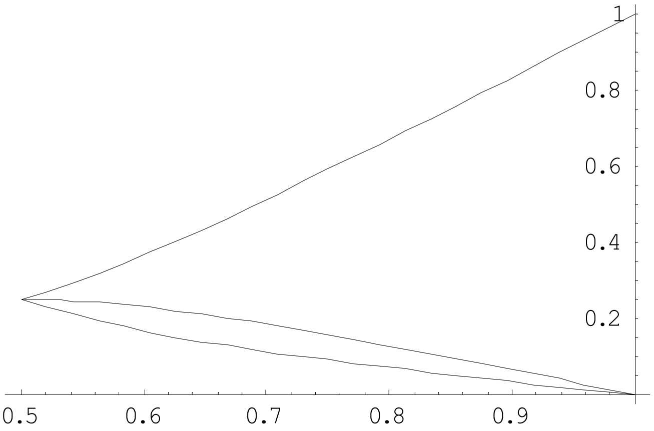
These constraints alone are not sufficient to derive the solution. Player 1 belongs to all credible coalitions, player 2 does not belong to 13, player 4 does not belong to either 12 or 13. (5) thus implies:

$$x_1^* = (1 + \mu_{13})x_2^* = (1 + \mu_{12})x_3^* = (1 + \mu_{12} + \mu_{13})x_4^*$$

Solving for μ_{12} and μ_{13} yields

$$\frac{1}{x_2^*} + \frac{1}{x_3^*} = \frac{1}{x_1^*} + \frac{1}{x_4^*}$$

This relationship can then be used to derive the payoff profile x^*. For $w_{12} = w_{13} = w > 1/2$, we plot in Figure 2 the values of $x_1^*, x_2^* = x_3^*$ and x_4^* as a function of w.



Convex games.

Some of the literature on coalition formation has considered convex games (see Chatterjee et al. (1993)). A game is *strictly convex* if the following property (PSC) holds:

> *PSC: For all* S, S' *such that* $S \cap S' \subsetneq S'$ *and* $S \cap S' \subsetneq S$, $v(S \cup S') > v(S) + v(S') - v(S \cap S')$.

It is well known that strictly convex games have a non-empty interior core. We have:

> **Claim A**: *When PSC holds, property P1 holds and the set of credible coalitions* $\mathcal{S}^* = \left\{S^{(1)}, ...S^{(m)}\right\}$ *must be nested, that is,* $S^{(1)} \subset S^{(2)} ... \subset S^{(m)} \subset N$.

This claim implies that strictly convex games have an asymptotically efficient equilibrium (by Proposition 1) and the profile of payoffs obtained in the limit coincides with the coalitional Nash bargaining solution (by Proposition 2).

The details of the proof appear in the appendix. The key observation (step 1 in the proof) is that if two coalitions S' and S'' are binding in the core allocation x the sense that $x(S') = v(S')$ and $x(S'') = v(S'')$ then either $S' \subseteq S''$ or $S'' \subseteq S'$. This is because otherwise due to the strict convexity either $S' \cap S''$ or $S' \cup S''$ would block x. This fact implies that in strictly convex games, there must exist at least one player who belongs to all credible coalitions, thereby guaranteeing that property P1 holds, and it also implies that the set of binding coalitions in the maximization of the Nash product must be nested.

Claim A also allows us to provide an algorithm for finding out the coalitional Nash bargaining solution of strictly convex games. Consider the sequence of coalitions $S^{(1)} \subset$

$S^{(2)} ... \subset S^{(m)}$ defined as follows:[10]

$$S^{(1)} \in \arg\max_{S} \frac{v(S)}{\mid S \mid}, \text{ and then } S^{(k+1)} \in \arg \max_{S, S \supsetneq S^{(k)}} \frac{v(S) - v(S^{(k)})}{\mid S \mid - \mid S^{(k)} \mid} \tag{7}$$

until $S^{(m)} = N$. Also let

$$u^{(1)} = \frac{v(S^{(1)})}{\mid S^{(1)} \mid} \text{ and for } k = 1, .., m-1, \; u^{(k+1)} = \frac{v(S^{(k+1)}) - v(S^{(k)})}{\mid S^{(k+1)} \mid - \mid S^{(k)} \mid} \tag{8}$$

> **Claim B**: *Consider a strictly convex game. Choose* m, $S^{(k)}$, *and* $u^{(k)}$, $k = 1, .., m$ *as defined above. The coalitional Nash bargaining solution allocation* x^* *is such that each player* i *in* $S^{(1)}$ gets $u^{(1)}$, *and each player* i *in* $S^{(k+1)} \backslash S^{(k)}$ gets $u^{(k+1)}$. *The set of credible coalitions is* $\mathcal{S}^* = \left\{ S^{(1)}, ...S^{(m-1)} \right\}$.

This claim follows from claim A, the observation that a player who belongs to more credible coalitions gets a higher payoff, and the conditions for an allocation to be in the core.

Buyer-submodular games. The recent literature on multi-object auctions (see Ausubel and Milgrom, 2002) has considered exchange problems between one seller and several buyers, and it has emphasized the role of the buyer-submodularity property (to be defined next) that ensures that the pivotal mechanism allocation (in which each buyer gets a payoff equal to the surplus he generates) lies in the core.

Formally, we assume that player 1 is the seller and players $i \neq 1$ are the buyers, so we have $v(S) > 0$ if and only if $1 \in S$. The function $v(.)$ is (strict) buyer-submodular if the following property holds:

> *PBS: For any* $S \subsetneq N$ and $i \in S$, with $i \neq 1$, $v(S) - v(S \setminus \{i\}) > v(N) - v(N \setminus \{i\})$.

We have:

> **Claim C**: *When PBS holds, property P1 holds and credible coalitions can only be of the form* $N \setminus \{i\}$, $i \neq 1$.

We provide a direct proof of this claim in the appendix. This claim is not surprising as we know from Ausubel Milgrom 2002 that for buyer-submodular games, the set of core allocations coincides with:

$$\bar{C}(v) \equiv \{x, \; x(N) = v(N) \text{ and } x_i \leq v(N) - v(N \setminus \{i\}) \text{ for all } i \neq 1\}$$

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[10] By strict convexity, this sequence is uniquely defined.

Finding the coalitional Nash bargaining solution thus reduces to finding a profile x satisfying

$$\begin{aligned} & \underset{x_2,\ldots x_n}{Max}\,(v(N)-\sum_{i\neq 1} x_i)\prod_{i\neq 1} x_i \\ x_i \;\; & \leq \;\; v(N)-v(N\setminus\{i\}) \text{ for all } i\neq 1 \end{aligned}$$

When all constraints are binding for the coalitional Nash bargaining solution x^*, then $x_i^* = v(N) - v(N \setminus \{i\})$ for all $i \neq 1$, and the solution coincides with the pivotal mechanism allocation,[11] which is the core allocation most preferred by the buyers. In general, the coalitional Nash bargaining solution need not coincide with the pivotal allocation. This is because in our bargaining protocol all players have the same chance of being the proposer, which equalizes the bargaining power of the seller and the buyers.

5 Analysis of the bargaining game.

We prove Propositions 1 and 2. Throughout the proof, for a given equilibrium, we denote by $q^{i,S}$ the probability that player i makes a proposal to S, and by $x^{i,S}$ the vector of payments that player i proposes to S, and by $u = (u_1, ..., u_n)$ the equilibrium value. We let $\underline{u}_i = \delta u_i$, and $\Delta = v(N) - \underline{u}(N)$. $\underline{u}_i$ corresponds to the threshold offer that player i would accept, and Δ to the equilibrium inefficiency induced by a rejection at the current date.[12] We let $\pi_i = \frac{1}{n}\sum_{j,S\in\widetilde{S}_i} q^{j,S}$. The probability π_i corresponds to the equilibrium probability that player i is not part of a coalition that forms (at the current date). We also let $\bar{u}_i$ denote the payoff that player i obtains in equilibrium when he is selected to make the proposal. Finally, we will refer to (P_Δ) as the program:

$$\arg\max_{x\in\mathcal{C}(\Delta)} \prod_{i\in N} x_i$$

and to x^*_Δ as its (unique solution).[13]

We start with Proposition 2.

Step 1: In equilibrium, if player i proposes to coalition S, it must be that he offers $\underline{u}_j = \delta u_j$ to player j, who accepts that offer. We note that since $v(N) > v(S)$ for all $S \neq N$,

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[11] This is the allocation in which each player $i \neq 1$ gets a payoff equal to the surplus he generates, that is, $v(N) - v(N \setminus \{i\})$.

[12] That is, as compared with the situation in which the grand team forms instantaneously.

[13] The solution is unique because $\mathcal{C}(\Delta)$ is convex and because the sets $\{x, \prod_{i\in N} x_i \geq \beta\}$ are strictly convex.

we must have $\underline{u}(N) \leq \delta v(N) < v(N)$. This implies that in equilibrium, when selected to move, players make proposals that are accepted with probability 1.

Step 2: There exists Δ_0 such that if $\Delta \leq \Delta_0$, then all players propose to the coalition N with positive probability.
Indeed, since the core has a non-empty interior, a proposal to a coalition $S \neq N$ (which would be accepted in equilibrium by step 1) would generate an inefficiency bounded away from 0, say ℓ. So if a player were to propose N with probability 0, $u(N)$ would be bounded away from $v(N)$ (by ℓ/n). Hence, if $\Delta \leq \ell/n$, it cannot be that a player proposes N with 0 probability.

Step 3: By step 2, in the event player i proposes, he gets a payoff $\bar{u}_i$ that satisfies $\bar{u}_i - \underline{u}_i = v(N) - \underline{u}(N)$. Besides, if $q^{i,S} > 0$, then $v(N) - \underline{u}(N) = v(S) - \underline{u}(S)$. By definition of Δ, we thus have that for all coalitions S that are proposed in equilibrium:

$$\underline{u}(S) = v(S) - \Delta.$$

Finally, if $i \in S$ and $q^{i,S} = 0$, then it must be $v(S) - \underline{u}(S) \leq v(N) - \underline{u}(N)$, as otherwise i would strictly prefer proposing to S.

It follows that for all S,

$$\underline{u}(S) \geq v(S) - \Delta.$$

Hence $\underline{u} \in \mathcal{C}(\Delta)$

Step 4: Equilibrium conditions can be written as follows:

$$u_i = \frac{1}{n}\bar{u}_i + \pi_i * 0 + (1 - \frac{1}{n} - \pi_i)\underline{u}_i$$

which, since $u_i = \frac{\underline{u}_i}{\delta}$ and $\bar{u}_i - \underline{u}_i = \Delta$, can re-written as:

$$(\frac{1-\delta}{\delta} + \pi_i)\underline{u}_i = \frac{1}{n}\Delta$$

Step 5: We must have $\underline{u} = x^*_\Delta$ Besides, for any $\Delta' < \Delta$, $\mathcal{N}(\Delta') > \mathcal{N}(\Delta)$.
This step results from the following Lemma, which we prove in the Appendix.

Lemma: *For any* $\Delta \geq 0$, let H_Δ denote the set of triplets (x, λ, η) *where* $\lambda = (\lambda_S)_S$ *with all* λ_S *non-negative and* $\eta \in \Re$ such that for some $a > 0$, the *following* *properties hold:*

$$\begin{aligned} x &\in \mathcal{C}(\Delta) && (9)\\ \lambda_S x(S) &= \lambda_S(v(S)-\Delta)) \text{ for all } S && (10)\\ (\lambda_N - \sum_{S\in\mathcal{S}_i}\lambda_S)x_i &= a, && (11)\\ a &= \prod_i x_i && (12)\\ \gamma &= \lambda_N - \sum_{S\in\mathcal{S}_i}\lambda_S && (13) \end{aligned}$$

Consider $(x,\lambda,\eta)\in H_\Delta$. *Then* $x = x^*_\Delta$. *Besides, if* $\gamma > 0$, then for any $\Delta' < \Delta$, $\mathcal{N}(\Delta') > \mathcal{N}(\Delta)$. And if $\gamma \leq 0$, then for any $\Delta' \geq \Delta$, $\mathcal{N}(\Delta') \geq \mathcal{N}(\Delta)$.

Equilibrium conditions derived in step 4 precisely allow us to find $\lambda,\gamma > 0$ such that $(\underline{u},\lambda,\gamma) \in H_\Delta$ (thereby showing step 5). Namely, set $\lambda_S = \frac{\delta}{1-\delta}\frac{1}{n}\sum_{j\in N} q^{j,S}$ and $\lambda_N = 1+\sum_S \lambda_S$. So it is immediate that $\gamma > 0$. Note that $\sum_{S\in\widetilde{\mathcal{S}}_i}\lambda_S = \frac{\delta}{1-\delta}\pi_i$. So by construction,

$$(\lambda_N - \sum_{S\in\mathcal{S}_i}\lambda_S)\underline{u}_i = (1+\frac{\delta}{1-\delta}\pi_i)\underline{u}_i = \frac{\delta}{1-\delta}\frac{1}{n}\Delta$$

so (11) holds with $a = \frac{\delta}{1-\delta}\frac{1}{n}\Delta$. Multiplying all λ_S and η by the same constant, we can also ensure that (12). Besides, for any S, either $q^{i,S} = 0$ for all i, in which case $\lambda_S = 0$, or there exists i such that $q^{i,S} > 0$, in which case $\underline{u}(S) = v(S) - \Delta$. So (10) holds. Finally, step 3 implies that $\underline{u} \in \mathcal{C}(\Delta)$. This concludes step 5.

Final Step. If one considers a sequence of discount factors δ^k tending to 1 and associated equilibrium values $u^{(k)}$ converging to an efficient allocation u^*, then there must exists a sequence $\Delta^{(k)}$ tending to 0 such that $\delta^{(k)}u^{(k)} = x^*_{\Delta^{(k)}}$, hence u^* must coincide with x^*, the coalitional Nash bargaining solution. Besides, since $\mathcal{N}(.)$ is continuous in Δ, and since $\mathcal{N}(\Delta^{(k)})$ is a strictly increasing sequence, Property $P1$ must hold.[14]

We now turn to Proposition 1. Note that Proposition 2 already implies that if property $P1$ does not hold then equilibrium outcomes with patient players must remain bounded away

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[14] Indeed, assume by contradiction that P1 does not hold. Then there must exist a sequence $\Delta_m \searrow 0$ such that $\mathcal{N}(\Delta_m) \geq \mathcal{N}(\Delta_{m+1})$. Since $\mathcal{N}(.)$ is continuous, we must have $\mathcal{N}(\Delta_m) \searrow \mathcal{N}(0)$. Now observe that for any $\Delta \in (0,\Delta_1)$, we must have $\mathcal{N}(\Delta) \geq \min(\mathcal{N}(\Delta_1),\mathcal{N}(0)) = \mathcal{N}(0)$. [This is because if $x^1 \in \mathcal{C}(\Delta_1)$ and $x^0 \in \mathcal{C}(0)$, then $y \equiv \alpha x^1 + (1-\alpha)x^0 \in \mathcal{C}(\alpha\Delta_1)$, and because $\prod_i y_i \geq \min\{\prod_i x_i^0, \prod_i x_i^1\}$]. So there could not exit a sequence $\Delta^{(k)} \searrow 0$ such that $\mathcal{N}(\Delta^{(k)}) \nearrow \mathcal{N}(0)$. Contradiction.

from efficiency. We now show that if $P1$ holds, then for patient enough players, we can construct equilibria with values arbitrarily close to the coalitional Nash bargaining solution.

Consider any $(x, \lambda, \gamma) \in H_\Delta$. By construction x corresponds to the solution of (P_Δ) and it is uniquely defined: $x = x^*_\Delta$. Since property $P1$ holds, there exists Δ_0 such that $\mathcal{N}(.)$ is strictly decreasing on $(0, \Delta_0)$, thus for any $\Delta < \Delta_0$ and $(x, \lambda, \gamma) \in H_\Delta$, we must have $\gamma > 0$. Also note that since $\mathcal{C}(\Delta)$ has a non-empty interior for Δ close enough to 0, all $x_i's$ are bounded away from 0, λ_N must be bounded (for each given Δ) and thus so are all the λ_S (since $\gamma > 0$).

For each $S \in \mathcal{S}$, either $\lambda_S = 0$, and we set $q^{i,S} = 0$ for all i; or $\lambda_S > 0$, and we set $q^{i,S} = \frac{n}{|S|} \frac{1-\delta}{\delta} \frac{\lambda_S}{\gamma}$ if $i \in S$.[15] Next define $\pi_i = \frac{1}{n} \sum_{j, S \in \widetilde{\mathcal{S}}_i} q^S_j$. We have:

$$\pi_i = \frac{1-\delta}{\delta} \sum_{S \in \widetilde{\mathcal{S}}_i} \sum_{j \in S} \frac{1}{n} q^{j,S} = \frac{1-\delta}{\delta} \sum_{S \in \widetilde{\mathcal{S}}_i} \frac{\lambda_S}{\gamma}$$

Thus

$$(\frac{1-\delta}{\delta} + \pi_i) x_i = \frac{1-\delta}{\delta} \frac{1}{\gamma} (\gamma + \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S) x_i = \frac{1-\delta}{\delta} \frac{1}{\gamma} (\lambda - \sum_{S \in \mathcal{S}_i} \lambda_S) x_i = \frac{1-\delta}{\delta} \frac{\mathcal{N}(\Delta)}{\gamma}.$$

Our aim will be to find Δ and $(x, \lambda, \gamma) \in H_\Delta$ such that

$$\frac{1-\delta}{\delta} \frac{\mathcal{N}(\Delta)}{\gamma} = \frac{1}{n} \Delta \tag{14}$$

For now, assume that we can find such a Δ. We consider the strategies where each player i chooses S with probability $q^{i,S}$, and threshold values are set to $\underline{u}_i = x_i$. We check below that these strategies constitute an equilibrium. We do that using the one-shot deviation principle. Assume that other players behave in this way, and when from next date on, player i also behaves in this way. We show that it is then optimal for player i to behave in this way today.

Indeed, by construction, for any i and S such that $q^{i,S} > 0$, $v(N) - x(N - \{i\}) = \Delta + x_i = v(S) - x(S - \{i\})$, so player i is indeed indifferent between choosing S or N. And for any i, S such that $q^{i,S} = 0$, $v(N) - x(N) = \Delta \geq v(S) - x(S)$, so not choosing S is optimal. Finally, computed from next date, player i's expected payoff is:

$$u_i = \frac{1}{n} (v(N) - x(N - \{i\}) + (1 - \frac{1}{n} - \pi_i) x_i.$$

It is thus optimal for player i to accept offers above δu_i. By construction we have:

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[15] Observe that $\sum_{S \neq N} q^{i,S}$ add up to less than 1 for δ sufficiently close to 1.

$$u_i - \frac{x_i}{\delta} = \frac{1}{n}(v(N) - x(N)) - (\frac{1-\delta}{\delta} + \pi_i)x_i = \frac{1}{n}(v(N) - x(N)) - \frac{\Delta}{n} = 0$$

It is thus optimal for player i to follow the proposed strategy.

It remains to show that one can find Δ that solves (14), and that such Δ tends to 0 when δ tends to 1.

For any $\Delta > 0$ and any $(x, \lambda, \gamma) \in H_\Delta$ let $h(\delta, x, \lambda, \gamma) = \min(\frac{n(1-\delta)\mathcal{N}(\Delta)}{\delta\gamma}, \Delta_0)$. Fix δ_0 small enough so that for all $(x, \lambda, \gamma) \in H_{\Delta_0}$ and $\delta \geq \delta_0$, $h(\delta, x, \lambda, \gamma) < \Delta_0$. Now fix any $\delta \geq \delta_0$, and choose $\underline{\Delta}$ small enough so that for all $(x, \lambda, \gamma) \in H_{\Delta_0}$, $h(\delta, x, \lambda, \gamma) > \underline{\Delta}$. The correspondence $\Delta \to h(H_\Delta, \delta)$ is non-empty convex, defined from $[\underline{\Delta}, \Delta_0]$ into itself, and it has a closed graph. Applying Kakutani fixed point theorem, it follows that there exists Δ^* and $(x, \lambda, \gamma) \in H_{\Delta^*}$ such that $h(\delta, x, \lambda, \gamma) = \Delta^*$. Since by construction Δ^* must be different from Δ_0, $h(\delta, x, \lambda, \gamma) < \Delta_0$, hence $h(\delta, x, \lambda, \gamma) = \frac{n(1-\delta)\mathcal{N}(\Delta^*)}{\delta\gamma} = \Delta^*$. So for any $\delta \geq \delta_0$, we can find $\Delta^*(\delta)$ and $(x, \lambda, \gamma) \in H_{\Delta^*}$ that solves (14), as desired.

Finally, consider any $\Delta \in (0, \Delta_0)$. For δ close enough to 1, it must be the case that $\frac{n(1-\delta)\mathcal{N}(\Delta')}{\delta\gamma} < \Delta'$ for all $\Delta' \geq \Delta$. So for δ small enough,. $\Delta^*(\delta)$ must be smaller than Δ. Since this is true for Δ arbitrarily small, $\Delta^*(\delta)$ must get close to 0 as δ tends to 1.

6 Discussion.

The role of $P1$.

It may be instructive to revisit a situation in which $P1$ does not hold to better understand why an efficient stationary equilibrium fails to exist in such a case.

To that end, consider example 2 above with $w = \frac{1}{2}$. In this limit case, there is a unique core allocation in which players 1 and 2 get $x = \frac{1}{2}$ and players 3, 4, 5 get $y = 0$. Consider now our non-cooperative bargaining game. We will explain below that in equilibrium, as δ gets close to 1, any player's payoff must remain bounded away from 0. This in turn will imply that asymptotically, the equilibrium value vector cannot be a core allocation, hence it cannot be efficient (by Proposition 0).

As explained in step 4, equilibrium conditions for player i can be written as

$$(\frac{1-\delta}{\delta} + \pi_i)\underline{u}_i = \frac{1}{n}(\bar{u}_i - \underline{u}_i) \tag{15}$$

where $\bar{u}_i$ is the payoff that player i obtains in the event he is selected to make a proposal. So one might expect to reduce $\underline{u}_i$ by having π_i large compared to $\frac{1-\delta}{\delta}$, that is, by increasing the probability that player i is not in the winning coalition. However, increasing π_i affects $\bar{u}_i - \underline{u}_i$. Indeed, let $\ell = \min_{S \subsetneq N}(v(N) - v(S))$ denote the minimum inefficiency associated with forming a coalition different from the grand team. We have:[16]

$$\bar{u}_i - \underline{u}_i \geq v(N) - \underline{u}(N) \geq v(N) - \delta(v(N) - \pi_i \ell),$$

which, combined with equality (15) implies that:

$$u_i = \frac{\underline{u}_i}{\delta} \geq \frac{1}{n}\frac{(1-\delta)v(N) + \delta\pi_i\ell}{(1-\delta) + \delta\pi_i} \geq \frac{\ell}{n},$$

hence the lower bound on player i's equilibrium value.

Intuitively, we can expect to reduce u_i by increasing the probability that player i is not in the winning coalition, but increasing that probability generates inefficiency losses that player i himself can extract once he gets to be the proposer.

Inefficient equilibria with $P1$.

The fact that $P1$ holds guarantees the existence of efficient equilibria when players are patient. Still, there may exist inefficient equilibria even if $P1$ holds. The 4-player example below illustrates this.

> **Example.** *Four players $i = 1, ..., 4$. We set $v(N) = 1$, $v(12) = x < 1$ and $v(13) = v(234) = y < 1$. We show below that if*[17]
>
> $$1 > x > \frac{6}{5}y \text{ and } \frac{1}{3}x + y > 1,$$
>
> *then there exists an asymptotically inefficient equilibrium with limit value u such that $u(N) = \frac{1}{2}(x+y)(< 1)$.*

We construct an equilibrium in which players 1 and 2 always propose to coalition $S = 12$, player 3 always proposes to coalition $S = 13$, and player 4 always proposes to coalition 234. We again let $\underline{u}_i$ denote the equilibrium acceptance threshold of player i. The probability π_i that player i is not in the winning coalition is 1/4 for players 1 and 2, 1/2 for player 3, and 3/4 for player 4. Simple adaptations of step 4 in Section 5 yield:

$$\left(\frac{1-\delta}{\delta} + \frac{1}{4}\right)\underline{u}_i = \frac{1}{4}(x - \underline{u}_1 - \underline{u}_2) \text{ for } i = 1, 2 \tag{16}$$

$$\left(\frac{1-\delta}{\delta} + \frac{1}{2}\right)\underline{u}_3 = \frac{1}{4}(y - \underline{u}_1 - \underline{u}_3) \tag{17}$$

$$\left(\frac{1-\delta}{\delta} + \frac{3}{4}\right)\underline{u}_4 = \frac{1}{4}(y - \underline{u}_2 - \underline{u}_3 - \underline{u}_4) \tag{18}$$

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[16] This is because when a coalition in which i does not belong forms (which happens with probability π_i) there is an efficiency loss at least equal to l.

[17] This set if not empty.

along with the inequalities:

$$x - \underline{u}_1 - \underline{u}_2 > y - \underline{u}_1 - \underline{u}_3 > y - \underline{u}_2 - \underline{u}_3 - \underline{u}_4 > 1 - \underline{u}_1 - \underline{u}_2 - \underline{u}_3 - \underline{u}_4 \tag{19}$$

Theses three inequalities ensure that it is optimal for players 1 and 2 to propose to coalition 12, for player 3 to propose to 13, and for player 4 to propose to 234 (rather than the grand coalition).

At the limit where δ tends to 1, the system of equations (16-18) yields a vector $\underline{u}$ such that:

$$\begin{aligned} \underline{u}_1 &= \underline{u}_2 = \frac{x}{3} \\ \underline{u}_3 &= \frac{1}{3}(y - \frac{x}{3}) \text{ and } \underline{u}_4 = \frac{1}{2}\underline{u}_3 \end{aligned}$$

The first inequality in (19) holds when $x > \frac{6}{5}y$, the second inequality in (19) always holds because $\underline{u}_1 = \underline{u}_2$, and the third inequality in (19) holds when $\frac{1}{3}x + y > 1$. Thus when both $x > \frac{6}{5}$ and $\frac{1}{3}x + y > 1$ hold, then for δ close enough to 1, the vector $\underline{u}^\delta = (\underline{u}_1^\delta, ..., \underline{u}_4^\delta)$ solution to (16-18) is arbitrarily close to $\underline{u}$ hence it also satisfies the inequalities (19). We have thus found an equilibrium, with acceptance threshold $\underline{u}^\delta$. Its limit value is $\underline{u}$ and satisfies $u(N) = \frac{1}{2}(x + y) < 1$.

7 Related literature

There is a vast literature on coalition formation. We discuss only a few insights of that literature and how they relate to/differ from the insights developed above.

Most models of coalition formation make the assumption that an agent who rejects a proposal becomes the agent to make a proposal in the next round (Selten (1981), Perry-Reny (1994), Moldovanu-Winter (1995), Bloch (1996), Chatterjee et al. (1993) etc). In the limit where there is no discounting, this feature directly implies that the profile of equilibrium payoffs of any stationary equilibrium must lie in the core as otherwise any member of a blocking coalition could reject a temptative equilibrium offer and propose in the next round an offer that would be accepted in equilibrium by the blocking coalition. Most papers in that literature (Chatterjee et al. (1993) and Okada (1996) are exceptions) consider scenarios without discounting in which case all core allocations can be sustained as equilibrium outcomes.

In our framework, there is no special role for the agent who rejects an offer, as every agent has the same probability of being the proposer whatever the history of play. This guarantees a stronger form of stationarity in our model, and it corresponds to an assumption also made in Okada (1996) as well as in the legislative bargaining literature (Baron and Ferejohn, 1987). Despite this difference, we find that a stationary equilibrium can only be efficient if the corresponding profile of equilibrium payoffs lies in the core. But, considering explicit discounting allows us to obtain a selection among the core allocations much in the same way as in the two-person bargaining problem in which Rubinstein (1982)'s bargaining model with discounting allows to select over those allocations that are Pareto-efficient and deliver each party more than his disagreement payoff.

For reasons expressed above, the papers closest to ours are Chatterjee et al. (1993) and Okada (1996) which also consider discounting. As already mentioned, Chatterjee et al. (1993) consider the case in which the first agent to reject the offer is the next proposer whereas Okada (1996) considers the variant in which the proposer is randomly picked in every period, as we do. Both papers ignore mixed strategies whereas they play a key role in our analysis.

In both Chaterjee et al. (1993) and Okada (1996), mixed strategies are ruled out. This may result in the inexistence of stationary equilibria. Chaterjee et al. (1993) provide conditions ensuring that a stationary equilibrium in pure strategies exist.[18]

Concerning efficiency results, Chatterjee et al. (1993) note in their setup that there may be situations with non-empty core and yet no efficient equilibria. Within our setup, we go further by providing a characterization of when this occurs (i.e. when property P1 does not hold).[19] Okada (1996) shows that efficiency requires that the egalitarian allocation $x^E = \left(x_i^E = \frac{v(N)}{n}\right)_{i \in N}$ lie in the core.[20] But this conclusion holds because of his restriction to *pure strategies*.[21]

Finally, Chatterjee et al. (1993) obtain a characterization of the equilibrium for convex games only, and the solution so obtained is not related to a simple optimization problem. In

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[18] Chatterjee et al. (1993) consider the class of no-delay equilibria (in which each player finds optimal to make a proposal). In that class, the fact that the first player to reject an offer is the next proposer ensures that one need not worry about the mixed strategies possibly employed by others to determine the acceptance thresholds of players.

[19] Inefficiencies may obtain in Chatterjee et al. (1993) for reasons that are unrelated to whether property P1 holds or not. It is more difficult to get efficiency in their setup (as compared to ours) because a player gains bargaining power by rejecting an offer: he becomes the next proposer.

[20] Within our setup, if the egalitarian allocation x^E in the core, then our analysis would predict that no subcoalition $S \subset N$ is credible, that the egalitarian allocation x^E coincides with the coalitional Nash bargaining solution, and that P1 holds.

[21] When x^E is not in the core, efficiency requires that some coalitions other than N are credible. Obtaining efficiency even when sub-coalitions are proposed requires that they are seldom proposed, hence the use of mixed strategies. This explains the difference of conclusion in Okada and our paper.

contrast our characterization holds for more general functions $v(\cdot)$, and the outcome of the bargaining game is related to a simple optimization problem, which also permits to identify which subcoalitions are credible as a function of $v(\cdot)$

On the positive side however, while (as in Selten's model) our game stops when one coalition forms, Chatterjee et al. and Okada consider the more challenging scenario in which the players left outside the coalition that forms can continue bargaining over the formation of a new coalition. Both assumptions are sensible depending on the application. In situations in which there is room for only one team (because subsequent teams, say, would get no surplus due to competition), our setup is more appropriate. In club good situations in which there is no rivalry between the forming teams, Chatterjee et al. or Okada's formulation is more appropriate. Clearly, more work is required to analyze within our present model the situation in which the left aside agents can continue bargaining after a coalition has formed.[22]

Some papers (see in particular, Hart and Mas-Colell (1996)) have provided some non-cooperative foundation to the Shapley value. In light of our insights that at most N coalitions can credibly form in equilibrium, it may seem surprising that some non-cooperative bargaining models might give rise to the Shapley value. As it turns out, these models all make an assumption that forces all coalitions to play a role in equilibrium. This is done by breaking the stationarity assumption made in our model. Specifically, in Hart and Mas-Colell the non-stationarity takes the form that with some positive (yet small) probability a proposer whose offer is rejected can no longer take part in the negotiation and receives 0 utility. By this assumption, all coalitions must play a role in equilibrium: in case of a rejection, there is always a chance that the other players have disappeared from the bargaining table, so the values of all coalitions must in turn affect the value to rejecting an offer, hence also the offers that are made in equilibrium.[23]

References

[1] Ausubel, L. and P. Milgrom (2002): 'Ascending auctions with package bidding,' *Frontiers of Theoretical Economics*, art. 1.

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[22] Observe that the insight that when there are k agents left, there are at most k coalitions which can credibly form still holds under this formulation.

[23] In Gul (1989) as in many papers of coalition formation (Konishi and Ray (2003) and Gomes and Jehiel (2005) are exceptions), it is never possible to undo a coalition, as once a coalition forms it is assumed that the coalition reduces to a single player. This creates non-stationarities and again it forces all coalitions to be possibly forming in equilibrium.

[2] Baron, D. and Ferejohn (1989), “Bargaining in Legislatures”, *American Political Science Review*, **90**, 316-330.

[3] Binmore, K., A. Rubinstein and A. Wolinsky (1986): ”The Nash Bargaining Solution”, *RAND Journal of Economics*, **17**, 176-188.

[4] Binmore, K., A. Shaked, and J. Sutton (1989): ’An outside option experiment,’ *Quarterly journal of economics*, **104**, 753-770.

[5] Bloch, F. (1996): ’Sequential formation of coalitions in games with externalities and fixed payoff division,’ *Games and Economic Behavior*, **14**, 90-123.

[6] Chatterjee, K., B. Dutta; D. Ray and K. Sengupta (1993): ’A Noncooperative Theory of Coalitional Bargaining,’ **60**, 463-47.

[7] Gomes, A and P. Jehiel (2005): "Dynamic Processes of Social and Economic Interactions: On the Persistence of Inefficiencies," *Journal of Political Economy* **113**, 626–667.

[8] Gul, F. (1989): ’Bargaining Foundations of Shapely Value,’ *Econometrica*, **57**, 81-95.

[9] Hart, S. and A. Mas-Colell (1996): ’Bargaining and Value,’ *Econometrica* **64**, 357-380

[10] Konishi, H. and D. Ray (2003): ’Coalition formation as a dynamic process,’ *Journal of Economic Theory* **110**,, 1-41

[11] Milgrom, P. (2007): ’Package Auctions and Package Exchanges’, *Econometrica*, **75**, 935-966.

[12] Moldovanu, B and E. Winter (1995): ’Order Independent Equilibria,’ *Games and Economic Behavior*, **95**, 21-35.

[13] Okada, A. (1996): ’A Noncooperative Coalitional Bargaining Game with Random Proposers,’ *Games and Economic Behavior*, **16**, 97-108.

[14] Perry, M. and P. Reny (1994): ’A Noncooperative View of Coalition Formation and the Core,’ *Econometrica*, **62**, 795-817.

[15] Rubinstein, A. (1982), “Perfect Equilibrium in a Bargaining Model”, *Econometrica* **50**, 97-109.

[16] Selten, R. (1981): ’A non-cooperative model of characteristic function bargaining,’ in Bohm and Nachtkamp (eds).

[17] Shapley, L. (1953): 'A value for n-person games', *Annals of Mathematical Studies*, **28**, 307-217.

Appendix.

Recall that we refer to (P_Δ) as the program:

$$\arg \max_{x\in\mathcal{C}(\Delta)} \prod_{i\in N} x_i$$

and to x^*_Δ as its (unique) solution. Since $\mathcal{C}(\Delta)$ has non-empty interior, the Kuhn and Tucker Theorem applies (see for example Theorem 28.3. in Rockafellar 1972). Define

$$L(x,\lambda,\Delta) = \prod_{i\in N} x_i + \sum \lambda_S(x(S) - (v(S)-\Delta)) + \lambda_N(v(N) - \Delta - x(N)).$$

A vector x is the solution to (P_Δ) if and only if there exists $\lambda \geq 0$ such that

$$\begin{aligned} x &\in \mathcal{C}(\Delta) && (20)\\ \lambda_S(v(S) - \Delta - x(S)) &= 0 && (21)\\ \frac{\partial L(x,\lambda)}{\partial x_i} &= 0 \text{ for all } i && (22) \end{aligned}$$

Condition (22) can be rewritten as:

$$\begin{aligned} (\lambda_N - \sum_{S\in\mathcal{S}_i} \lambda_S)x_i &= a && (23)\\ \text{with } a &= \prod_{i\in N} x_i, && (24) \end{aligned}$$

Proof of the Lemma: If we can find $a > 0, \lambda, x$ that satisfy (20),(21) and (23), then multiplying λ by a constant, we can also ensure that we find λ, x that satisfy (24) as well, hence x must be the solution to (P_Δ). Finally, we have:

$$L(x,\lambda,\Delta') = L(x,\lambda,\Delta) + (\lambda_N - \sum_S \lambda_S)(\Delta - \Delta').$$

Since $\mathcal{N}(\Delta) = \min_\lambda \max_x L(x,\lambda,\Delta)$, we get that if $\gamma = \lambda_N - \sum_S \lambda_S > 0$, then for all $\Delta' < \Delta$, $\mathcal{N}(\Delta') \geq \mathcal{N}(\Delta) + \gamma(\Delta - \Delta')$; and if $\gamma \leq (<)0$, then for all $\Delta' \geq \Delta$, $\mathcal{N}(\Delta') \geq (>)\mathcal{N}(\Delta)$.

Proof of Proposition 3: Consider the program (P_0) and x^*_0 its unique solution. Condition (P1) implies that $\lambda_N - \sum_{S\in\mathcal{S}} \lambda_S \geq 0$, because otherwise, there would exists $\Delta_0 > 0$ such that $\mathcal{N}(\Delta) > \mathcal{N}(0)$ for all $\Delta \in (0,\Delta_0)$, hence, since $\mathcal{N}(\Delta)$ is continuous, $(P1)$ could not hold.

We distinguish two cases. If $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S > 0$, then we set $\eta = \lambda_N - \sum_{S \in \mathcal{S}} \lambda_S$ and $\mu_S = \lambda_S / \eta$. Then, since

$$\begin{aligned}
\lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S &= \lambda_N - \sum_{S \in \mathcal{S}} \lambda_S + \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S \\
&= \eta(1 + \sum_{S \in \widetilde{\mathcal{S}}_i} \mu_S)
\end{aligned}$$

equations (23) imply the desired equality:

$$(\alpha + m_i^{\mu})x_i = (\alpha + m_1^{\mu})x_1 \text{ for all } i. \tag{25}$$

with $\alpha = 1$.

If $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S = 0$, then we define $\mu_S = \lambda_S$ and obtain:

$$\lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S = \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S = \sum_{S \in \widetilde{\mathcal{S}}_i} \mu_S$$

hence equations (23) now imply the desired equality (25) with $\alpha = 0$.

Proof of claim A:

Step 1. Consider a strictly convex game and a core allocation[24] x such that $x(S') = v(S')$ and $x(S'') = v(S'')$ for $S' \neq S''$ and S', S'' both different from N. Then either $S' \subset S''$ or $S'' \subset S'$.

Proof of step 1. Assume by contradiction that $S' \cap S'' \subsetneq S'$ and $S' \cap S'' \subsetneq S''$, Then,[25]

$$\begin{aligned}
x^*(S' \cup S'') &= x^*(S') + x^*(S'') - x^*(S' \cap S'') \\
&= v(S') + v(S'') - x^*(S' \cap S'') \\
&\leq v(S') + v(S'') - v(S' \cap S'') \\
&< v(S' \cup S'')
\end{aligned}$$

which contradicts the fact that $x^* \in \mathcal{C}(v)$. So either $S' \subset S''$ or $S'' \subset S'$. **Q. E. D.**

Step 2. There is a player who belongs to all credible coalitions. This follows from step 1.

Step 3. Property P1 is satisfied. This follows from step 2 and the remark (see example 1, footnote 6) that when there is a key player, P1 must hold.

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[24] It is well known that strictly convex games have a non-empty core.

[25] The first inequality follows from $x^*(S' \cap S'') \geq v(S' \cap S'')$ because x^* is a core allocation. The second inequality follows from strict convexity.

Step 4. The set of credible coalitions is an increasing sequence $S^{(1)} \subset S^{(2)} ... \subset S^{(m)} \subset N$.

Proof of claim B:

Step 1. Consider the coalitional Nash bargaining solution, say u^*, and let $\{S^{(1)}, S^{(2)}, ..., S^{(m)}\}$ be the set of credible coalitions. From claim A, $S^{(1)} \subset S^{(2)} ... \subset S^{(m)}$, and from Proposition 3, all players in $S^{(1)}$ receive the same payoff, say $u^{(1)}$; for $k = 1, ...m - 1$, all players in $S^{(k+1)} \setminus S^{(k)}$ receive the same payoff, say $u^{(k+1)}$.

Step 2. Together with the conditions $x(S^{(k)}) = v(S^{(k)})$ for $k = 1, ...m$ and $x(N) = v(N)$, Proposition 3 implies that for all k, $u^{(k)}$ is a strictly decreasing sequence.

Step 3. Since the coalitional Nash bargaining solution u^* is a core allocation, (7) must hold. Indeed, steps 1 and 2 imply that for all $i \in N$, $u_i^* \leq \frac{v(S^{(1)})}{|S^{(1)}|}$. If there existed S such that $\frac{v(S)}{|S|} > \frac{v(S^{(1)})}{|S^{(1)}|}$, then the allocation u^* would not be a core allocation. So we must have $S^{(1)} \in \arg\max_S \frac{v(S)}{|S|}$. Similarly, for all $i \in N \setminus S^{(k)}$, $u_i^* \leq \frac{v(S^{(k+1)}) - v(S^{(k)})}{|S^{(k+1)}| - |S^{(k)}|}$. If there existed $S \supsetneq S^{(k)}$ such that $\frac{v(S) - v(S^{(k)})}{|S| - |S^{(k)}|} > \frac{v(S^{(k+1)}) - v(S^{(k)})}{|S^{(k+1)}| - |S^{(k)}|}$, then the allocation u^* would not be a core allocation. So $S^{(k+1)} \in \arg\max_{S, S \supsetneq S^{(k)}} \frac{v(S) - v(S^{(k)})}{|S| - |S^{(k)}|}$.**Q. E. D.**

Proof of Claim C. Property P1 holds because player 1 must belong to all credible coalitions. Let x^* be the coalitional Nash bargaining solution and consider $S \subsetneq N$. Assume by contradiction that $S \setminus \{i\}$ is credible. This implies that $x^*(S \setminus \{i\}) = v(S \setminus \{i\})$. Since $x^*(N) = v(N)$, and since $x^*(N) + x^*(S \setminus \{i\}) = x^*(N \setminus \{i\}) + x^*(S)$, we obtain,

$$x^*(N \setminus \{i\}) + x^*(S) = v(N) + v(S \setminus \{i\})$$

hence, using property (PBS):

$$x^*(N \setminus \{i\}) + x^*(S) < v(N \setminus \{i\}) + v(S)$$

The latter inequality implies that either $v(N \setminus \{i\}) > x^*(N \setminus \{i\})$ or $v(S) > x^*(S)$, contradicting the fact that x^* is a core allocation. **Q. E. D.**