The winner's curse with private values.[*]

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Abstract

This paper challenges the view that winner's curse phenomena should be attributed to the existence of an element commonly valued by the bidders. We show that winner's curse phenomena may arise in private value setting. We also show that bid functions may exhibit properties that are generally thought to be inconsistent with the private value paradigm, namely that in first and second price auctions, bid functions may become less agressive when the number of bidders rise.

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1 Introduction

In some competitive environments, \successful" bidders are not so successful after all: \successful" bidders (that is, those who won the competition) tend to obtain returns that (on average) lie below initial projections. This discrepancy between realized and anticipated returns, and the possibility that winning bidders end up making losses, has been called the winner's curse.[1]

In the context of competition for oil ¯eld lease, Capen and al. (1971) have given the following explanation for the winner's curse. The quantity of reserves in a given track is unknown to bidders, and each only has imperfect estimates of the true track value. Thus each bidder may either overestimate true tract value, or underestimate it. Capen and al. (1971) then note that

> \in competitive bidding, the winner tends to be the player who most overestimates true tract value.... [So] a player tends to win a biased set of tracks - namely those on which he has over estimated value or reserves".

As a result, even when a bidder's evaluations are correct on average, a bidder' evaluations on the tract he wins are not correct on average: they are biased upward.

In the context of procurement, Milgrom (1989) has re-formulated the same idea. Consider a contract to be auctioned to bidders who all have the same cost of doing the job and assume each bidder has an imperfect estimate of the cost of doing the job. Each bidder may either overestimate true cost, or underestimate it. In competitive bidding, the winning bidder tends to be the one who most underestimate true cost. Thus even when a contractor's estimates of costs are correct on average, a contractor's estimates of costs on the contracts he wins are not correct on average: they tend to be biased downward.

In both examples, bidders are uncertain about true valuation or cost, and competition induces a simple selection bias in favor of the most optimistic ones. Bidders who would fail to take this selection bias into account at the bidding stage would be subject to the winner's curse.

A seemingly key aspect of the two stories given above is that there exists a true track value - respectively a true cost- which is common to all bidders. How crucial is the

[1] The terminology is due to Capen and al. (1971).

common value assumption?

The ¯rst part of this paper is devoted to showing that the common value assumption is not important. Speci¯cally, we show that Capen and al. insight extends to the private value setting where valuations (or costs) are drawn from independent distribution (rather than being perfectly correlated) and where bidders are uncertain about their own valuation.[2] The intuition is straightforward. Consider the procurement context. Now assume that the contractors's costs are drawn from independent distribution and that each observes an imperfect estimate of his own cost. It will turn out that the same logic applies:

> In competitive bidding, a bidder is more likely to win when he underestimates his own costs. So a bidder tends to win a biased set of contracts - namely those on which he underestimates his own cost.

It will then follow that even when a contractor's estimates of costs are correct on average, a contractor's estimates of costs on the contracts he wins are not correct on average: they tend to be biased downward.

How then should reasonably sophisticated bidders behave? What advice could we give bidders? It is standard to claim that one stays immune to the winner's curse by bidding cautiously. Milgrom (1989) for example suggests the following:

> To make money in competitive bidding, you will need to mark up your bids twice: once to correct for the underestimation of costs on the projects you win, and a second time to include a margin for pro¯ts.

Besides, since it is reasonable to expect the selection bias to increase when competition gets ¯ercer, he adds the following:

> The markup to adjust for underestimation will have to be larger the larger is the numbers of your competitors.

These advice are generally thought to be sensible in common value situations. Yet our previous conclusion that Capen and al. insight extends to private value setting

[2]Note that the terminology private value often refers to the fact that bidders know their valuation. We ¯nd more natural to de¯ne the private value case (independent draws) by opposition with the common value case (perfectly correlated draws). See also the discussion page 15 and footnote 18.

suggest that they should be sensible more generally. The purpose of Section 3 is to show that indeed these advises are sensible even when one considers a private value setting (where valuations or costs are drawn from independent distribution). More speci¯cally, we consider bidders who bid a constant mark-up over their estimate, and who look for the optimal mark-up. In a second price auction, for example, we ¯nd that in a symmetric setting, the equilibrium mark-up increases with the number of bidders. In other words, bidders become less aggressive when competition rises.

Our conclusions may be puzzling to some. First, the theoretical literature that has followed the work of Capen and al. supports the view that the winner's curse may only arise in a common (or correlated) value setting. Second, in a private value setting, equilibrium analysis of a second price auction suggests that bidding behavior should be independent of the number of bidders, seemingly contradicting the result described above. The purpose of section 4 will be to clarify these puzzles. Section 5 will then discuss the various assumptions of our model.

For now, let us point out that theoretical models provide an explanation for the winner's curse that di®ers sharply from (and is more sophisticated than) that suggested by Capen and al.

To account for the winner's curse, the theoretical literature typically makes two assumptions:

(i) the object for sale has a characteristic that is identically valued by the bidders, and

(ii) bidders are di®erentially informed about the value of this characteristic.

These two assumptions imply that other bidders' estimates are correlated with own value. As a result information concerning other bidders' estimate may improve a bidder's assessment of his own valuation. Winning an auction conveys some information about other bidders bids - namely that others bidders' bids are lower than one's own bid- hence also about other bidders' estimates (because bidders use their estimate to choose their bid). The winner's curse arises when bidders fail to take into account the information on others' estimates conveyed by winning.

In the latter explanation, the common value assumption is key because without it, other bidders' estimate would not be correlated with a bidder's own valuation, hence winning would convey no information on own valuation, thus failure to take into account information on others' estimates would not be harmful. In contrast, Capen and al. 's view is that competition induces a selection bias in favor of optimistic bidders. This selection bias exists (so we will show) whether valuations are drawn from independent distributions or not. And failure to take into account this selection bias is potentially harmful to bidders.

2 The winner's curse in a private value setting

Let us start with a simple illustration, which builds on the example given in Milgrom (1989). We will first describe in details the common value setting discussed by Milgrom (1989), and we will then move to the private value setting.

Consider the problem of bidding for construction contracts. There are n potential contractors bidding for a construction contract. Each contractor $i = 1, \ldots, n$ does not know precisely what the job will cost, and he regards the actual cost C_i of the job as a random variable. For simplicity, it is assumed that each bidder has an unbiased estimate

$$X_i = C_i + \varepsilon_i$$

of C_i, where ε_i is an estimation error. We will shortly make some assumption regarding the joint distribution over costs. For now we assume that each (marginal) distribution over C_i is non-degenerate, admits a density f_i with support $[\underline{c}_i, \bar{c}_i]$. The errors ε_i, $i = 1, \ldots, n$ are assumed to be independent across bidders and independent of costs. For simplicity, we assume that they are drawn from identical (and non-degenerate) distributions having a density g with support $[\underline{\varepsilon}, \bar{\varepsilon}]$ where $\underline{\varepsilon} < 0 < \bar{\varepsilon}$. We will also assume that f_i and g are positive on $(\underline{c}_i, \bar{c}_i)$ and $(\underline{\varepsilon}, \bar{\varepsilon})$ respectively.

Common costs In the common value setting, the actual cost C_i is the same for all bidders. We shall denote by C the random variable defining this common cost.[3] Suppose that all bidders determine their bids b_i by using the same rule (that is, $b_i = b(X_i)$ for some function b assumed to be strictly increasing).[4] Then the winning bidder is the one with the lowest estimate. Let $\underline{X}_{-i}$ denote the lowest estimate among bidders other than

[3] For any realization c of C, we have $C_i = c$ for all i.

[4] In this section, we do not make any assumption concerning bidder's behavior, except for the fact that they use the same bidding function. We will analyze bidder's behavior in the next Section.

i. Conditional on winning, the expected cost is thus equal to

$$E[C \mid X_i \le \underline{X}_{-i}].$$

Even though each contractor's individual estimate X_i is an unbiased estimate of the actual cost C, this estimate X_i is a biased estimate of the cost conditional on winning. Indeed, let $\underline{\varepsilon}_{-i} = \min_{j \ne i} \varepsilon_j$. We have[5]

$$E[C - X_i \mid X_i \le \underline{X}_{-i}] = -E[\varepsilon_i \mid \varepsilon_i \le \underline{\varepsilon}_{-i}] > 0.$$

In other words, a bidder who would use X_i to predict future costs would be disappointed on average: his actual cost would on average exceed his prediction. The difference above measures the winner's curse in the common value setting, and for future reference we denote it Δ^c.

In a first price auction for example, if all bidders bid by adding a constant markup a over their estimate, that is, if

$$b(X_i) = X_i + a$$

and if they naively expect to obtain a profit equal to a in case they win, they will be disappointed.. On average, their profit (conditional on winning) will only be equal to $a - \Delta^c$. In particular, if $a < \Delta^c$, winners incur losses.

Private costs The key observation of this paper, on which we will later elaborate, is that the analysis above does not hinge on the common cost assumption. In what follows, we consider a private value setting: we assume that individual costs C_i are drawn from independent distribution. The expected difference between actual cost and estimate, conditional upon winning, is a measure of the winner's curse, and it is denoted Δ_i. That is, we define

$$\Delta_i \equiv E[C_i - X_i \mid X_i \le \underline{X}_{-i}]. \tag{1}$$

As for the common value setting, the estimate X_i is a biased estimate of costs upon winning (unless bidder i wins with probability 1). Formally we have:

Proposition 1 Assume $1 > \Pr\{X_i \le \underline{X}_{-i}\} > 0$. Then $\Delta_i > 0$.

[5] The inequality holds because (i) for any $y \ge \underline{\varepsilon}$, $E[\varepsilon_i \mid \varepsilon_i \le y] \le 0$, (ii) for any $y \in [\underline{\varepsilon}, 0]$, $E[\varepsilon_i \mid \varepsilon_i \le y] < 0$ (since ε_i is non degenerate), and (iii) $\Pr\{\varepsilon_i \le \underline{\varepsilon}_i \le 0\} > 0$.

Indeed, we have

$$E[C_i - X_i \mid X_i \leq \underline{X}_{-i}] = E[-\varepsilon_i \mid X_i \leq \underline{X}_{-i}].$$

For any realizations $\underline{x}_{-i}; c_i$ such that $\underline{x}_{-i} - c_i \leq \underline{\varepsilon}$, bidder wins with probability 0, and for any realizations $\underline{x}_{-i}; c_i$ such that $\underline{x}_{-i} - c_i > \underline{\varepsilon}$,

$$E[\varepsilon_i \mid X_i \leq \underline{X}_{-i}; C_i = c_i; \underline{X}_{-i} = \underline{x}_{-i}] = E[\varepsilon_i \mid \varepsilon_i \leq \underline{x}_{-i} - c_i]: \qquad (2)$$

The right-end side of Equation (2) is non positive, and strictly negative whenever $\underline{x}_{-i} < c_i + \bar{\varepsilon}$. Since the event $\{X_i \leq \underline{X}_{-i} < C_i + \bar{\varepsilon}\}$ has positive probability (otherwise bidder i would be winning with probability either 1 or 0),[6] we obtain the desired inequality.

The winner's curse phenomenon is thus not specific to common value settings. Winning the auction induces a selection bias: Even in a private value setting, a bidder is more likely to win when he under-estimates costs. If a bidder does not take into account this selection bias, he will be disappointed on average when the actual cost is realized: the actual cost will turn out to be, on average, higher than his initial estimate.

We conclude this Section with some comparative statics and a numerical example.

Some comparative statics The next proposition derives the effect of increased competition on the selection bias.

In what follows, we restrict our attention to the symmetric case where the costs C_i, $i = 1; \ldots; n$, are all drawn (independently) from the same distribution f, and where the errors ε_i, $i = 1; \ldots; n$, are all drawn (independently) from the same distribution f_0. We assume that f_0 is positive on its support $[\underline{\varepsilon}; \bar{\varepsilon}]$, and that f is positive on its support $[\underline{c}; \bar{c}]$. Let $\underline{x} = \underline{c} + \underline{\varepsilon}$ and $\bar{x} = \bar{c} + \bar{\varepsilon}$. For any $x \in [\underline{x}; \bar{x}]$, we define:

$$Y_i(x) = E[C_i \mid X_i = x]$$

We have:

Proposition 2 (i) for all n, $\Delta_i \leq |\underline{\varepsilon}|$
(ii) $\lim_{n \nearrow \infty} \Delta_i = |\underline{\varepsilon}|$
(iii) Assume $Y_i(x) - x$ is decreasing. Then Δ_i increases with the number of bidders.

[6] Note that herre, we use the assumption that f_i and g are positive on their support.

Proposition 2 shows that the selection bias tends to its highest possible level when the number of bidders gets very large. To interpret (iii), observe that the difference $Y_i(x) - x$ represents how optimistic bidder i is when $X_i = x$ (because when $Y_i(X_i) > X_i$, bidder i underestimates expected cost). (iii) states that if bidders are more optimistic when they have a lower estimate X_i, then the selection bias Δ_i increases with the number of bidders. The intuition is that in the event where bidder i wins, bidder i tends to have a lower estimate when the number of bidders is larger (because low realizations of $\underline{X}_{-i}$ are more likely), hence he tends to be more optimistic.

A numerical example. We have explained above that competition induces a selection bias whether a private or a common value setting is considered. Although the same qualitative effect obtains in both cases, it might be much stronger in a common value setting than in a private value setting.

We provide a simple numerical example, with the aim of comparing the selection biases depending on whether costs are private or common.[7] Specifically, for the private value setting, we assume that

$$C_i = c^0 + \varepsilon_i$$

with ε_i and ε_j drawn independently from the same distribution with density f. And for the common value setting, we assume that

$$C = c^0 + \varepsilon$$

with ε drawn from that same distribution, with density f.[8] Computations are made for the following density:

$$f(z) = (\frac{1}{2} - z)(\frac{1}{2} + z) \text{ for } z \in [-\frac{1}{2}, \frac{1}{2}], \text{ and } f(z) = 0 \text{ otherwise.}$$

The following table gives the value of Δ_i, Δ^c and Δ_i/Δ^c as a function of the number of bidders. It shows that even when the number of bidders is small, the ratio Δ_i/Δ^c is

[7] Note that at the limit when n is very large, Δ_i and Δ^c both tend to $|\underline{\varepsilon}|$. So the aim of the example is to compare the values of Δ_i and Δ^c when the number of bidders is not large.

[8] These assumptions ensure that the marginal distributions over the estimate X_i are the same in both settings (private and common), making the comparison between the two settings a relevent one.

significant.

# of bidders	2	3	4	5	6	7	8	9	10
Δ_i	0.09	0.14	0.16	0.18	0.20	0.21	0.22	0.23	0.24
Δ^c	0.13	0.19	0.23	0.26	0.28	0.30	0.31	0.32	0.33
Δ_i/Δ	0.7	0.7	0.7	0.71	0.71	0.72	0.72	0.72	0.73

3 The bid functions.

How then should reasonably sophisticated bidders behave? What advice could we give them? Rephrasing Milgrom (1989), an advise that seems reasonable is the following: To be immune from the winner's curse, bidders should bid cautiously and add a markup to their estimate.[9] The purpose of this Section is to formalize this idea and find out how the optimal markup varies when competition gets fiercer.

Throughout this section, we will assume that bidders correct the bias induced by competition by adding some fixed mark-up to their estimate. That is, we assume that bidders use a bidding function of the form[10]

$$b_i : X_i \to X_i + a_i,$$

and choose the mark-up a_i optimally. This assumption will be discussed in the next Section. Our objective here is to assess how the optimal mark-up varies with competition.

We consider two auction formats: The first price auction in which the bidder whose bid is lowest gets the contract at a price equal to his bid; and the second price auction, in which the bidder whose bid is lowest gets the contract at a price equal to the next lowest bid.

We restrict again our attention to symmetric cases: in the private cost case, costs (respectively errors) are drawn independently from the same distribution with density f (respectively with density f') across players. We denote by h (and respectively $\underline{h}$) the induced distribution of (say) bidder i's estimate X_i (respectively of the lowest estimate

[9] In a first price auction for example, one should add a markup to generate profits. This markup should not be too large so that chances of winning are not compromised. But it should certainly not be too low, because it might be that winning will just be the result of too much optimism

[10] The additive specification is made for simplicity. Other specifications such as the multiplicative specification would yield similar insights. (See the Discussion and the Appendix).

$\underline{X}_{-i}$ among bidders other than i). In the common cost case, C is also drawn from a distribution with density f.

For any given auction format, we can compute the expected profit $G(a_i; a^*)$ that bidder i obtains when he chooses the markup a_i and the rest of the bidders choose $a_j = a^*$.[11] A symmetric solution (or Nash equilibrium - see Section 4 for a discussion of the solution concept) is a markup a^* that satisfies:

$$G(a^*; a^*) \geq \max_{a_i} G(a_i; a^*). \tag{3}$$

We start with the analysis of the second price auction.

Second price auction. If bidder i chooses a markup a_i while other bidders use a markup a^*, bidder i wins when $X_i + a_i < \underline{X}_{-i} + a^*$ and then makes a profit equal to $\underline{b}_i - C_i = \underline{X}_{-i} - X_i + a^* + \varepsilon_i$. Bidder i's expected profit is thus equal to:

$$G(a_i; a^*) = \Pr\{X_i + a_i < \underline{X}_{-i} + a^*\} \times E[\underline{X}_{-i} - X_i + a^* + \varepsilon_i \mid X_i + a_i < \underline{X}_{-i} + a^*]$$

We have:[12]

Proposition 3 In the second price auction, if a^* is a solution to (3) it satisfies:

$$a^* = E[-\varepsilon_i \mid X_i = \underline{X}_{-i}]$$

The intuition is straightforward. Assume for example that all bidders choose a markup $\hat{a} > a^* = E[-\varepsilon_i \mid X_i = \underline{X}_{-i}]$. Then if bidder i decides to use a markup equal to $\hat{a} - \epsilon$ rather than $\hat{a}$, with $\epsilon > 0$ very small, the allocation changes only in the events where $X_i \in [\underline{X}_{-i}; \underline{X}_{-i} + \epsilon]$. In such events, bidder i does not obtain the object if he chooses the markup $\hat{a}$, and he obtains an expected gain approximately equal to equal to $E[\underline{X}_{-i} - X_i + \hat{a} + \varepsilon_i \mid X_i = \underline{X}_{-i}] = \hat{a} - a^* > 0$ if he chooses the markup $a^* - \epsilon$.

The markup a^* can be interpreted as a selection-bias mark-up. When bidder i competes with other bidders, his estimate X_i is a biased estimate of his cost upon winning.

[11] Note that to compute $G(a_i; a^*)$, we take the expectation over all realizations of X_i. The interpretation is that each bidder i looks for a markup a_i that is optimal on average, across all possible realizations of X_i.

[12] Note that Proposition 5 gives a necessary condition satisfied by any solution to (3). This condition is obtained by checking first order conditions. Existence of a symmetric solution requires additional conditions on the distributions.

The markup a^* captures how a sophisticated bidder (aware of the winner's curse) should correct his estimate.

The following Proposition explains how this mark-up is affected by an increase in competition (that is, a rise in the number of bidders):[13]

Proposition 4 (a) $\lim_{n \to \infty} E[-\theta_i \mid X_i = \underline{X}_{-i}] = -\underline{\varepsilon}$
(b) If $\lambda(x) = Y_i(x) - x$ decreases with x, then $E[-\theta_i \mid X_i = \underline{X}_{-i}]$ increases with the number of bidders.

So when competition increases, the selection bias mark-up tends to $-\underline{\varepsilon}$, its highest possible level. To interpret (b), recall that the difference $Y_i(x) - x$ represents how optimistic bidder i is when $X_i = x$ (because when $Y_i(X_i) > X_i$, bidder i underestimates expected cost). Proposition 4 (b) states that if bidders are more optimistic when they have a lower estimate X_i, then the selection bias mark-up increases with the number of bidders.[14]

Finally, it is instructive to see in our numerical example how the equilibrium mark-up differs depending on whether we consider the common or private cost setting. Note that Proposition 3 above does not actually use the assumption that costs are drawn from independent distributions (see Appendix). In case costs are identical across bidders, the condition $X_i = \underline{X}_{-i}$ becomes $\theta_i = \underline{\theta}_{-i}$, and in the second price auction, the equilibrium mark-up in the common cost case, which we denote a_c^*, satisfies:

$$a_c^* = E[-\theta_i \mid \theta_i = \underline{\theta}_{-i}]$$

The following table gives (as a function of the number of bidders) the equilibrium mark-up in the private cost case (first line), the equilibrium mark-up in the common cost case

[13] Recall that $Y_i(x) = E[C_i \mid X_i = x]$.

[14] This assumption is for example satisfied in our numerical example: since γ_i and ε_i are drawn from the same distribution,

$$Y_i(x) - x = E[-\theta_i \mid C_i + \theta_i = x] = E[-\theta_i \mid \gamma_i + \theta_i = x - c^0] = -\frac{x - c^0}{2};$$

hence $\lambda'(x) = -1/2 < 0$.

(second line), and a comparison of the order of magnitude these two mark-ups.

# of bidders	2	3	4	5	6	7	8	9	10
a^*	0	0.06	0.09	0.12	0.14	0.15	0.17	0.18	0.19
a^*_c	0	0.10	0.15	0.19	0.22	0.24	0.26	0.27	0.28
a^*/a^*_c	—	0.59	0.60	0.62	0.63	0.63	0.64	0.65	0.65

It thus appears in this example that the way bidders respond to increased competition does not depend much on whether bidders estimates are statistically independent (as in the private value case) or correlated (as in the common value case).

This result is important in light of the empirical literature on auctions which is concerned with distinguishing between common and private value settings (see for example Paarsch (1992), Laffont and Vuong (1996) and Hendricks et al. (1999); see also a survey by Laffont (1997)). It suggests that one should be cautious in interpreting the data: the empirical finding that bid functions would become less aggressive with a rise of the number of bidders may not be an evidence that bidders face a common value setting.

First price auction. By choosing a_i when the rest of the bidders choose $a_j = a^*$, bidder i wins when $X_i + a_i < \underline{X}_{-i} + a^*$ and obtains a profit equal to $b_i - C_i = \varepsilon_i + a_i$ in that case. Thus we have:

$$G(a_i; a^*) = \Pr\{X_i + a_i < \underline{X}_{-i} + a^*\} \times E[\varepsilon_i + a_i \mid X_i + a_i < \underline{X}_{-i} + a^*]$$

The next Proposition characterizes the symmetric solution. Define

$$\phi(x) = \Pr\{X_i < \underline{X}_{-i} + x\}$$

We have:

Proposition 5 In the first price auction, if a^* is a solution to (3) then it satisfies:

$$a^* = \frac{\phi(0)}{\phi'(0)} + E[-\varepsilon_i \mid X_i = \underline{X}_{-i}]$$

So compared to the second price auction, bidders have to increase further their markup by an amount equal to $\frac{\phi(0)}{\phi'(0)}$. This term can be interpreted as a strategic markup, and it corresponds to the markup bidders would choose if X_i were truly equal to the cost C_i (this would be the case if the errors ε_i, $i = 1, \ldots, n$ were degenerate - and equal to 0): bidder i would want to increase his bid above X_i to make positive profits in case of winning, but only to a reasonable extent because otherwise he would compromise his chances of getting the contract.

When competition increases, the strategic mark-up tends to 0:

Proposition 6 $\lim_{n \nearrow \infty} \frac{\phi(0)}{\phi'(0)} = 0$

Proposition 4 and 6 say nothing about the overall effect of increased competition on bidders aggressiveness: does the equilibrium mark-up increase or decrease with competition? There is no general answer to this question: depending on the form of the distributions, bidding functions may become either more aggressive and less aggressive with the number of bidders. The following proposition however provides a condition under which the limit mark-up tends to $-\underline{\varepsilon}$ from below as the number of bidders becomes very large. Under this condition, the increase in the selection-bias mark-up eventually dominates the decrease in the strategic mark-up.

Proposition 7 Assume $(\lambda + \frac{H}{h})'$ is negative at $\underline{x}$, and that for x close to $\underline{x}$, $h(x) \geq c(x - \underline{x})^m$ for some constants $c > 0$ and $m \geq 0$. Then the limit mark-up tends to $-\underline{\varepsilon}$ from below.

Whether the condition of Proposition 7 holds or not depends on the tail of the distribution h. In our numerical example, the condition is satisfied: $h(x)$ tends to 0 at $\underline{x}$ like $(x - \underline{x})^3$, so $(\frac{H}{h})'(\underline{x}) = \frac{1}{4} < \frac{1}{2} = -\lambda'(\underline{x})$.

4 A comparison with standard results

Our results seem to be at odds with the theoretical literature that has followed the work of Capen and al.. First, this literature supports the view that the winner's curse may only arise in a common (or correlated) value setting. Second, it shows that private and common value settings generate fundamentally different behaviors, in particular with respect to the effect of increased competition. Yet in our analysis, these two settings appear to yield very similar behavior from the bidders.[15]

The purpose of this Section is to explain why our findings differ from standard ones.

[15] More precisely, in a private value setting, the traditional approach predicts that (i) in a second price auction, bidding functions are independent of the number of bidders, and (ii) in a first price auction,

The winner's curse in a private value setting. As mentioned in the introduction, the theoretical models that have followed the early work of Capen and al. provide an explanation for the winner's curse that differs from that of Capen and al. In these models, other bidders estimates are correlated with own valuation, and the winner's curse arise if and only if bidders fail to take into account the information on other's estimate conveyed by winning. Without correlation between others' estimates and own valuation, there can be no winner's curse.

Implicit in these models is the assumption that for each possible value of their estimate X_i, bidders can make appropriate inferences concerning expected costs. Formally, whenever bidder i obtains an estimate $X_i = x$, he is able to compute the conditional expectation of costs

$$Y_i(x) = E[C_i \mid X_i = x].$$

When bidder i correctly predicts expected costs, then there can be no winner's curse in a private value setting, because

$$Y_i(x) = E[C_i \mid X_i = x] = E[C_i \mid X_i = x; X_i < \underline{X}_{-i}]$$

Thus in the theoretical models, bidders are neither optimistic, nor pessimistic: they are just correct about the way they predict expected costs.

In Capen and al., as well as the informal stories that we usually give to illustrate the winner's curse phenomenon, bidders make estimation errors: they are sometimes too optimistic about their costs, meaning that

$$X_i < Y_i(X_i)$$

and they are sometimes too pessimistic about their costs, meaning that

$$X_i > Y_i(X_i).$$

Implicit in these analysis, as well as in our Section 1, is the assumption that bidders do not know the distributions from which costs and errors are drawn, or that they are unable to compute the conditional expectation function $Y_i(x)$.

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bidders bid more aggressively when competition increases. To the contrary we find that (in private value settings), as competition increases, bidders bid (unambiguously) less aggressively in the second price auction, and possibly less aggressively in the first price auction.

Bidding behavior. There are a priori many bid functions $b_i(X_i)$ that a bidder might consider using. In Section 3, we have assumed that bidders look for a bidding function that is optimal in a restricted class of bidding functions, namely, bidding functions of the form

$$b_i(X_i) = X_i + a_i.$$

Other functional forms such as

$$b_i(X_i) = \beta(X_i; a_i) \text{ with } \beta(X_i; 0) = X_i, \text{ and } \beta_1', \beta_2' > 0$$

would have given us similar insights (see Appendix).

We are implicitly making two assumptions:

(i) bidders are restricted to looking for an optimal bidding function in a given (one-dimensional) family of bidding functions, that we parameterized by the scalar a_i.[16]

(ii) The scalar a_i parameterizes a uniform change with respect to the identity function (under which a bidder would bid his estimate X_i).[17]

We now wish to explain why each of these two assumptions is important.

(i) Assume that bidders have access to the true conditional expectation of cost $Y_i(x)$, and that they look for a bidding function that is optimal in the following class:

$$b_i : X_i \to Y_i(X_i) + a_i.$$

Then it is easy to check that in the second price auction, the optimal markup is equal to 0, and that in the first price auction $a^* = \frac{\psi(0)}{\psi'(0)}$, where $\psi(x) = \Pr\{Y_i(X_i) < \min_{j \neq i} Y_j(X_j) + x\}$. Comparative statics with respect to the number of bidders would not differ from standard theoretical models: in the second price auction, bidding behavior would be unaffected by the number of bidders; and in the first price auction, bidders would become more aggressive as competition gets fiercer.

(ii) Assume that bidders are not restricted to a particular class of function (this is the standard theoretical model). Then in the second price auction for example, it is

[16] Our solution concept can thus be interpreted as a constrained Nash equilibrium where bidders' strategies are restricted to a given (one-dimensional) family of bidding functions.

[17] a_i is normalized to 0 when bids coincide with estimates.

standard to prove that the optimal bid function is equal to $Y_i(X_i)$, independently of the number of bidders.

To summarize, the distinction between the estimate X_i and the conditional expectation $Y_i = Y_i(X_i)$ is a key ingredient of our analysis.

In section 2, it explains the winner's curse because a player is more likely to win when $Y_i > X_i$. And this possibility arises in both common and private value setting.

In section 3, it also explains why bidders who anticipate the winner's curse would bid more cautiously as competition increases: when competition rises, the value of the estimate X_i at which bidders tend to win gets smaller, and for such low values of X_i, the difference $Y_i(X_i) - X_i$ tends to be large (See Figure 1). In other words, winners happen to be on average more optimistic when competition is fiercer. To compensate for this greater optimism, (sophisticated) bidders need to increase their markup.

In standard models, this distinction between the estimate X_i and the conditional expectation Y_i is irrelevant: any function of X_i, say $b(X_i)$, can be viewed as a function $b \circ Y_i^{-1}$ of Y_i. As a consequence, theoretical models usually identify the estimate with the conditional expectation given the estimate, and when bidders are not restricted to choosing a bid function in a particular class of functions, there is no loss of generality in doing so.[18,19]

5 Discussion and extensions

One virtue of the model we presented is that it closely captures the intuition (most, I believe) economists have in mind when informally discussing the winner's curse. One possible criticism of our model though is that the assumptions seem to depart sharply

[18] Note that in private value models that assume risk neutrality, the conditional expectation is further identified with the true valuation, which is why these models assume that bidders know their valuations.

[19] There are actually early works on competitive bidding in which restrictions on the strategies used by bidders have been imposed. (See Rothkopf (1969,1980), and also Paarsch (1992) who presents various models where simple bidding strategies such as the multiplicative or the additive strategy are considered). It may thus seem surprising that these papers did not come up with insights similar to ours. The reason is that, when dealing with private value setting, one usually assumes that each bidder knows how much the object for sale is worth to him (see footnote 18. And one then considers strategies that are simple functions of a bidder's own valuation. In contrast, we have considered strategies that are simple function of their own estimate.

from standard Bayesian models:

(i) Bidders are implicitly assumed to be unaware of distributions of costs and errors (or unable to compute conditional expectations);

(ii) Bidders look for an optimal bidding function in a limited set of bid functions.

In addition, there seem to be a contradiction between these two assumptions: How can bidders find the optimal markup is they do not know the distributions over costs and errors? The purpose of this section is to address these criticisms.

The prediction that bidders consistently make losses when they win an auction does not seem to be a reasonable. Economists usually predict that in the long run, bidders should learn from past experience, correct past mistakes and end up anticipating the winner's curse.[20]

More generally, we (as many economists) hold the view that equilibrium behavior should be interpreted as the outcome of a learning process in which bidders who repeatedly face similar situations attempt to adjust their strategies to improve their gains. Rest points of the learning process, that is, points at which neither bidder find a way to improve his gains by choosing an alternative strategy, coincide with equilibrium points.

An important question however is the following: what is the set of alternatives that a bidder considers choosing from? This issue is generally not addressed in simple games. In the context of auctions, the set of possible functions is very large. In the absence of restrictions on the strategies considered, there is little hope that bidders learn to play a best response (let alone equilibrium behavior).[21]

In our view, restrictions on the set of functions should be interpreted as a device to facilitate learning: When bidder i looks for an optimal bid function in the set $\{b; b : X_i \to \beta(X_i; a); a \in \Re\}$, he is looking for a parameter a^* that is optimal not for a particular realization of X_i, but on average across the various possible realizations of X_i. So in effect, bidder is pooling information from all past informational nodes $X_i^{t'}$, $t' < t$, whether the current estimate X_i^t differs from $X_i^{t'}$ or not. In other words, for the purpose

[20] See however Kagel and Levin (1986), who question whether bidders are sophisticated enough to even anticipate the winner's curse.

[21] Bidders would have to learn the optimal bid for each possible realization of the estimate, which requires that they accumulate experience [that is, information on past bids, past realized costs upon winning, past prices] for each possible realization of the estimate.

of finding a good markup a^*, bidder i is considering all the possible informational nodes as analogous. Yet he realizes that when he bids, the particular realization of X_i matters a great deal: so the bid function depends on X_i.

With this view, the two assumptions mentioned above are not in contradiction. Bidders do not know distributions over costs and errors. But this does not prevent them from using past observations to improve the choice of the markup.

To conclude, let us mention one alternative interpretation or extension of our model. We have so far emphasized the role of learning, thereby justifying that bidders need not have prior knowledge of the distributions over costs and errors. One alternative view is that bidders do have priors, yet possibly incorrect ones. In this context, bidder i would find reasonable to compute the conditional expectation

$$\hat{\beta}_i(X_i) = E_i[C_i \mid X_i].$$

where the notation E_i refers to the fact that the expectation is taken given bidder i's priors. In the context of a second price auction, bidder i would presumably find reasonable to choose to bid $\hat{\beta}_i(X_i)$. At the same time, bidder i may realize that his priors may not be correct and that correcting his bid to

$$b(X_i) = \hat{\beta}_i(X_i) + a_i$$

might be a better idea. How the correction a_i should be chosen depends on past experience: over time, as experience will accumulate, bidder i should be in position to improve the choice of the correcting term, and learn to choose the markup a_i optimally (in expectation, given $\hat{\beta}_i$, the true (unknown) distribution over costs and errors, and other bidders' behavior).

In the case where bidder i's prior g_i happens to coincide with the true distribution over own cost and estimate, bidder i should learn to choose $a_i = 0$, independently of the number of bidders. In other cases, bidder i's optimal bid will depend on the number of bidders and their behavior.

More generally, our analysis suggests a way to deal with situations where (i) bidders do not have correct (or common) priors and (ii) bidders are aware of this and attempt to correct their decision, based on experience. Because the correction is based on experience, the true distributions matter [e.g. the correction a_i is optimal on average, given the true distributions]. But because/if bidders' ability to correct their decision is limited [e.g constraint on the set of bidding functions considered], priors matter too. The application of this idea to other Bayesian games is left for future research.

Appendix

Proof of Proposition 2: We denote by h (respectively $\underline{h}$) the distribution over x_i (respectively $\underline{x}_{-i}$) induced by f and f^0. We also let

$$H(y) \equiv \Pr\{X_i \leq y\} = \int_{x \leq y} h(x)dx \text{ and } \underline{H}(y) \equiv \Pr\{\underline{X}_{-i} \geq y\} = (1 - H(y))^n.$$

And we define the functions

$$\phi : y \to E[-\theta_i \mid C_i + \theta_i \leq y] \text{ and}$$

$$\psi : y \to \frac{\int_{\underline{x}}^{y} H(x)\underline{h}(x)dx}{\int_{\underline{x}}^{\bar{x}} H(x)\underline{h}(x)dx}$$

Since f and f^0 are continuous densities, the functions ϕ and ψ are continuously differentiable on $[\underline{x}; \bar{x}]$. We let M be a bound on ϕ^0 (Note that this bound is independent of n).

By definition of Δ_i, we have:

$$\Delta_i = \frac{\int_{\underline{x}}^{\bar{x}} \phi(y)H(y)\underline{h}(y)dy}{\int_{\underline{x}}^{\bar{x}} H(x)\underline{h}(x)dx} = \int_{\underline{x}}^{\bar{x}} \phi(y)\psi^0(y)dy \tag{4}$$

Integrating by part the right-hand side of equation (4), we have

$$\Delta_i - \phi(\underline{x}) = \int_{\underline{x}}^{\bar{x}} \phi^0(y)(1 - \psi(y))dy \tag{5}$$

To prove (ii), we show that when n gets large, the right-hand side of equation (5) gets close to 0. To prove (iii), we observe that $\phi^0 \leq 0$, (because $Y_i(x) - x$ is decreasing), and that for any y, $(1 - \psi(y))$ decreases with n.

(ii) Note that under our symmetry assumption,

$$\int_{\underline{x}}^{\bar{x}} H(x)\underline{h}(x)dx = \Pr\{X_i \leq \underline{X}_{-i}\} = 1/n: \tag{6}$$

This implies (since $H(x) \leq 1$) that

$$1 - \psi(y) \leq n\int_{y}^{\bar{x}} H(x)\underline{h}(x)dx \leq n\underline{H}(y) = n(1 - H(y))^n;$$

which further implies that for any $\varepsilon > 0$,

$$\begin{aligned} |\Delta_i - \phi(\underline{x})| &\le \int_{\underline{x}}^{\underline{x}+\varepsilon} \left|\phi'(y)\right| dy + \int_{\underline{x}+\varepsilon}^{\bar{x}} \left|\phi'(y)\right| (1-\psi(y))dy \\ &\le M(\varepsilon + n(1-H(\underline{x}+\varepsilon))^n(\bar{x}-\underline{x})). \end{aligned}$$

For any fixed $\varepsilon > 0$, there exists n_0 large enough so that for all $n > n_0$,

$$n(1-H(\underline{x}+\varepsilon))^n(\bar{x}-\underline{x}) \le v.$$

Since ε can be chosen close to 0, we get that Δ_i gets close to $\phi(\underline{x})$ when n gets large.

(iii) We have $\underline{h}(y) = (n-1)h(y)(1-H(y))^{n-2}$. Using (6), we rewrite $1-\psi(y)$ as:

$$\begin{aligned} 1-\psi(y) &= n(n-1)\left[\int_y^{\bar{x}} h(x)(1-H(x))^{n-2}dx - \int_y^{\bar{x}} h(x)(1-H(x))^{n-1}dx\right] \\ &= n(1-H(y))^{n-1} - (n-1)(1-H(y))^n \end{aligned}$$

For any $z \in (0,1)$, , the function $\lambda(s) = sz^{s-1} - (s-1)z^s$ decreases with s when $s \ge 1$.[22] So for any y, $1-\psi(y)$ decreases with the number of bidders. ■

Proof of Proposition 5: Let k denote the density of the joint distribution over $X_i, \underline{X}_{-i}$. In the case of independent distributions, $k(y,z) = h(y)\underline{h}(z)$. We have

$$\phi(x) \equiv \Pr\{X_i < \underline{X}_{-i} + x\} = \int_{\underline{x}}^{\bar{x}} \left(\int_{\underline{x}}^{z+x} k(y,z)dy\right)dz$$

We define:

$$\lambda(y) = E[-\theta_i \mid C_i + \theta_i = y]$$

We have, by definition of G:

$$G(a_i, a^*) = \phi(a^* - a_i)a_i - \int_{\underline{x}}^{\bar{x}} \left(\int_{\underline{x}}^{z+a^*-a_i} \lambda(y)k(y,z)dy\right) dz$$

which gives:

$$\left.\frac{\partial G}{\partial a_i}\right|_{a_i=a^*} = -\phi'(0)a^* + \phi(0) + \int_{\underline{x}}^{\bar{x}} \lambda(z)k(z,z)dz$$

Since $\phi'(0) = \int_{\underline{x}}^{\bar{x}} k(z,z)dz$, the last term is equal to $\phi'(0)E[-\theta_i \mid X_i = \underline{X}_{-i}]$. So the first order condition $\left.\frac{\partial G}{\partial a_i}\right|_{a_i=a^*} = 0$ is satisfied if $a^* = \frac{\phi(0)}{\phi'(0)} + E[-\theta_i \mid X_i = \underline{X}_{-i}]$, as desired. ■

[22] Observe that $\lambda'(s) = z^{s-1}[1-z+(s(1-z)+z)\ln z]$. When $s \ge 1$ and $z \in (0,1)$, $(s(1-z)+z) > 1$. Hence, since $\ln z < 0$, $\lambda'(s) < z^{s-1}[1-z+\ln z] < 0$.

Lemma 1 Consider any function g defined on $[\underline{x},\bar{x}]$. Assume that g is continuous on $[\underline{x},\bar{x}]$ and positive on $(\underline{x},\bar{x})$. For any $s \geq 0$ and $x \in [\underline{x},\bar{x}]$, define

$$B(s,x) = \frac{\int_x^{\bar{x}} g(z)(1-H(z))^s dz}{\int_{\underline{x}}^{\bar{x}} g(z)(1-H(z))^s dz}$$

(i) For any $x > \underline{x}$ and any $\varepsilon > 0$, there exists n_0 such that for all $s \geq n_0$, $B(s,x) \leq \varepsilon$.
(ii) For any $x > \underline{x}$ and $s \geq 1$, $\frac{\partial B}{\partial s} < 0$.

Proof. (i) Choose any $\varepsilon > 0$. For any $x > \underline{x}$, there exists $x_0 \in (\underline{x},x)$ such that $1 - H(x_0) = 2(1 - H(x))$. Let $G = \int_x^{\bar{x}} g(z)dz$ and $G_0 = \int_{\underline{x}}^{x_0} g(z)dz$. We have:

$$B(s,x) \leq \frac{\int_x^{\bar{x}} g(z)(\frac{1-H(x_0)}{2})^s dz}{\int_{\underline{x}}^{x_0} g(z)(1-H(x_0))^s dz} = \frac{(G-G_0)}{2^s G_0}. \tag{7}$$

For s sufficiently large, the right-hand side is below ε.

(ii) For any $s > 0$, the condition $\frac{\partial B}{\partial s} < 0$ is equivalent to

$$\frac{\int_x^{\bar{x}} \ln(1-H(z)).g(z)(1-H(z))^{s-1}dz}{\int_x^{\bar{x}} g(z)(1-H(z))^s dz} < \frac{\int_{\underline{x}}^{\bar{x}} \ln(1-H(z)).g(z)(1-H(z))^{s-1}dz}{\int_{\underline{x}}^{\bar{x}} g(z)(1-H(z))^s dz}$$

which is itself equivalent to

$$\frac{\int_x^{\bar{x}} \frac{\ln(1-H(z))}{1-H(z)} g(z)(1-H(z))^s dz}{\int_x^{\bar{x}} g(z)(1-H(z))^s dz} < \frac{\int_{\underline{x}}^{x} \frac{\ln(1-H(z))}{1-H(z)} g(z)(1-H(z))^s dz}{\int_{\underline{x}}^{x} g(z)(1-H(z))^s dz}: \tag{8}$$

Since $y \mapsto \frac{\ln(1-y)}{1-y}$ decreases with y (on $(0,1)$), for any z, z' such that $z > x > z'$, we have:

$$\frac{\ln(1-H(z))}{1-H(z)} < \frac{\ln(1-H(x))}{1-H(x)} < \frac{\ln(1-H(z'))}{1-H(z')}$$

Hence inequality (8) holds. ■

Lemma 1 has the following corollary:

Corollary 1 Consider any function g defined as in Lemma 1, and any function ϕ continuously differentiable on $[\underline{x},\bar{x})$. For any $n \geq 0$, define .

$$A(n) = \frac{\int_{\underline{x}}^{\bar{x}} \phi(z)g(z)(1-H(z))^n dz}{\int_{\underline{x}}^{\bar{x}} g(z)(1-H(z))^n dz}$$

(i) For any $\eta > 0$, there exists n_0 such that for all $n \geq n_0$, $A(n) \geq \phi(\underline{x}) - \eta$.
(ii) If $\phi' < 0$, then for any $n \geq 0$, $A(n+1) \geq A(n)$.
(iii) If $\phi'(\underline{x}) < 0$ and if $\lim_{z \searrow \underline{x}} g(z)/(z-\underline{x})^m > 0$ for some $m > 0$, then for n_0 large enough, $A(n) \leq \lim_{n' \to \infty} A(n')$ for all $n \geq n_0$.

Proof. Define $B(s;x)$ as in Lemma 1.

(i) Choose x_0 such that $\eta(x) \geq \eta(\underline{x}) - \nu/2$ for $x \in [\underline{x}; x_0]$, and let M be an upper bound on $|\eta|$. We have:

$$A(n) \geq (\eta(\underline{x}) - \nu/2)(1 - B(n;x_0)) - MB(n;x_0)$$

The result (i) follows because by Lemma 1, for n large enough $B(n;x_0)$ is as small as we want.

(ii) We have

$$A(n) = \int_{\underline{x}}^{\bar{x}} \eta(z)\frac{\partial B}{\partial x}(n;z)dz \tag{9}$$

Integrating the right-hand side of (9) by part, we get:

$$A(n) = \eta(\underline{x}) + \int_{\underline{x}}^{\bar{x}} \eta'(z)B(n;z)dz$$

The result (ii) then follow immediately from the property (ii) of B derived in Lemma 1

(iii) Choose $x_0 > \underline{x}$ such that $\eta'(x) < -a$ for all $x \in [\underline{x}; x_0]$, for some $a > 0$, and let M be an upper bound on $|\eta|$. Since on $[\underline{x}; x_0]$, $\eta(z) \leq \eta(\underline{x}) - a(z - \underline{x})$, we have

$$A(n) - \eta(\underline{x}) \leq -a\int_{\underline{x}}^{x_0}(z - \underline{x})\frac{\partial B}{\partial x}(n;z)dz + M(\bar{x} - x_0)B(n;x_0)$$

There are two terms on the right-hand side. We now show that the second one tends to 0 much faster than the first one, as n gets very large. The second term tends to 0 faster than 2^{-n} (see inequality (??)) in the proof of Lemma 1. For the first term, let x_n be such that $H(x_n) = \frac{1}{n}$. Note that since h is bounded, $x_n - \underline{x} \geq \frac{k}{n}$ for some k. Since $\lim_{z \searrow \underline{x}} g(z)/(z - \underline{x})^m > 0$, we have , for some constant b independent of n,

$$\int_{\underline{x}}^{x_0}(z - \underline{x})\frac{\partial B}{\partial x}(n;z)dz \geq b(1 - \frac{1}{n})^n\int_{\underline{x}}^{x_n}(z - \underline{x})^{m+1}dz:$$

So the first term does not tend to 0 faster than $n^{-(m+2)}(>> 2^{-n})$. ■

Proof of Proposition 4, 6 and 7: We have

$$\frac{\phi(0)}{\phi'(0)} = \frac{\int_{\underline{x}}^{\bar{x}} H(z)\underline{h}^{(n-1)}(z)dz}{\int_{\underline{x}}^{\bar{x}} h(z)\underline{h}^{(n-1)}(z)dz} = \frac{\int_{\underline{x}}^{\bar{x}} \frac{H(z)}{h(z)}(h(z))^2(1 - H(z))^{n-2}dz}{\int_{\underline{x}}^{\bar{x}}(h(z))^2(1 - H(z))^{n-2}dz}$$

Choose $g = h^2$ and $\eta = \frac{H}{h}$. Since $g > 0$ and since $\eta(\underline{x}) = 0$, Corollary 1 implies $\lim_{n \nearrow \infty} \frac{\phi(0)}{\phi'(0)} = 0$. Similarly, we have:

$$E[-\theta_i \mid X_i = \underline{X}_i] = \frac{\int_{\underline{x}}^{\bar{x}} \lambda(z)h(z)\underline{h}^{(n-1)}(z)dz}{\int_{\underline{x}}^{\bar{x}} h(z)\underline{h}^{(n-1)}(z)dz} = \frac{\int_{\underline{x}}^{\bar{x}} \lambda(z)(h(z))^2(1 - H(z))^{n-2}dz}{\int_{\underline{x}}^{\bar{x}}(h(z))^2(1 - H(z))^{n-2}dz}$$

Choose $g = h^2$. Since $g > 0$ and $\lambda(\underline{x}) = -\underline{\epsilon}$, applying Corollary 1 yields $\lim_{n \to \infty} E[-\epsilon_i \mid X_i = \underline{X}_{-i}] = -\underline{\epsilon}$. If in addition $\lambda' > 0$, then Corollary 1 (ii) also implies Proposition 6.

Finally, to prove Proposition 7, observe that with n bidders, the equilibrium mark-up is equal to

$$a^{(n)} = \frac{\int_{\underline{x}}^{\bar{x}} \eta(z)\,(h(z))^2\,(1 - H(z))^{n-2}\,dz}{\int_{\underline{x}}^{\bar{x}} (h(z))^2\,(1 - H(z))^{n-2}\,dz}$$

with $\eta = \lambda + \frac{H}{h}$. Under the assumption of Proposition 7, $\eta'(\underline{x}) < 0$. The result therefore follows from Corollary 1 (iii). ■

Proof of Proposition 3: Using the notation used in the proof of Proposition 5, we have, by definition of G:

$$G(a_i; a^*) = \int_{\underline{x}}^{\bar{x}} \left(\int_{\underline{x}}^{z + a^* - a_i} (z - y + a^* - \lambda(y))k(y; z)dy \right) dz$$

which gives:

$$\left.\frac{\partial G}{\partial a_i}\right|_{a_i = a^*} = \int_{\underline{x}}^{\bar{x}} (\lambda(z) - a^*)k(z; z)dz$$

So the first order condition $\left.\frac{\partial G}{\partial a_i}\right|_{a_i = a^*} = 0$ is satisfied if $a^* = E[-\epsilon_i \mid X_i = \underline{X}_{-i}]$, as desired. ■

To conclude this Appendix, we briefly explain how our analysis can be extended to more general bid functions. Consider bid functions of the form

$$\beta(X; a)$$

satisfying

$$\begin{aligned} \beta(X; 0) &= X \\ \beta_1' &> 0; \beta_2' > 0 \end{aligned}$$

We show the following:

In a second price auction, any symmetric equilibrium mark-up a^* satisfies:

$$E\left[\frac{\beta_2'(X_i; a^*)}{\beta_1'(X_i; a^*)} (\beta(X_i; a^*) - X_i) \,\middle|\, X_i = \underline{X}_{-i} \right] = -E\left[\frac{\beta_2'(X_i; a^*)}{\beta_1'(X_i; a^*)} \epsilon_i \,\middle|\, X_i = \underline{X}_{-i} \right]$$

Indeed, let γ be the inverse bid function, that is:

$$\gamma(\beta(X; a); a) = X.$$

Using the notation used in the proof of Proposition 5, the function G becomes:

$$G(a_i; a^*) = \int_{\underline{x}}^{\bar{x}} \left(\int_{\underline{x}}^{\gamma(\beta(z;a^*);a_i)} (\beta(y;a^*) - y - \lambda(y))k(y;z)dy \right) dz$$

which gives, since $\gamma_2'(\beta(z;a^*);a^*) = -\frac{\beta_2'(z;a^*)}{\beta_1'(z;a^*)}$:

$$\left.\frac{\partial G}{\partial a_i}\right|_{a_i=a^*} = -\int_{\underline{x}}^{\bar{x}} \frac{\beta_2'(z;a^*)}{\beta_1'(z;a^*)} (\beta(z;a^*) - z - \lambda(z))\, k(z;z)dz$$

Remark 1 In case of a multiplicative bid function

$$\beta(X;a) = (1+a)X$$

we obtain, since $\beta_1'(X_i;a) = 1 + a$ and $\beta_2'(X_i;a) = X$,

$$a^* = -\frac{E[X_i \epsilon_i | X_i = \underline{X}_i]}{[(X_i)^2 | X_i = \underline{X}_i]}$$

Remark 2 When the number of bidders gets very large, the equilibrium mark-up tends to a^∞ satisfying

$$\beta(\underline{x}; a^\infty) - \underline{x} = |\underline{\epsilon}|$$

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