

# Pooling Cherries and Lemons

Some simple economics of complex financial products\*

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## Abstract

We study banks' incentives to create financial products backed by assets of heterogeneous quality assuming that investors assess such products by extrapolating from small and possibly different samples of the underlying assets. A monopolistic bank chooses heterogeneous bundles when investors have enough money so as to exploit the resulting dispersion of their assessments. When the number of banks is sufficiently large, oligopolistic banks opt for full bundling even when there is little money and even if this turns out to be collectively detrimental to the banks, which we refer to as the Bundler's Dilemma.

**Keywords:** complex financial products, bounded rationality, disagreement, market efficiency.

**JEL codes:** C72; D53; G14; G21

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# 1 Introduction

Many financial products such as mutual fund shares or asset-backed securities consist of claims on composite pools of assets. An exact assessment of the value of such products may require analyzing each asset in the underlying pool. But, this is not an easy task, especially for novel products on which investors have little prior knowledge. As an (imperfect) alternative, investors may instead identify the value of the pool with their assessment of a sample of assets in the pool, as if the sample were representative of the entire pool. This valuation method can be related to the representativeness heuristic as well as the extrapolative heuristic, which have been widely discussed in psychology as well as in the context of financial markets.<sup>1</sup>

Given that (at least some) investors would use such valuation method to assess composite assets, we ask ourselves: What are the incentives for banks or other financial institutions to package assets into composite pools? What is the effect of such bundling on asset prices? Are banks' strategies likely to be different depending on whether there is more or less money in the hands of potential investors? Are their incentives influenced by market competition in the banking system? Addressing such questions is essential -we believe- for the large debate on the increasing complexity of financial products.<sup>2</sup>

We develop a simple model to address these questions. For concreteness, consider several banks holding assets (say, loan contracts) of different quality (say, probability of default). Banks are able to package their assets into pools as they wish and sell (pass-through) securities backed by these pools. Our key assumption is that investors have a limited ability to assess the value of the pools, which we describe as follows: Each investor randomly samples one asset from each pool (we discuss briefly the case of larger samples in an extension) and assumes that the average value of the assets in the pool coincides with this draw considered as "representative." If an investor samples an asset with expected value  $\tilde{x}$  from pool  $\alpha$ ,

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<sup>1</sup>Tversky and Kahneman (1975) discuss the representativeness heuristic and Tversky and Kahneman (1971) introduce the "law of small numbers" whereby "*people regard a sample randomly drawn from a population as highly representative, that is, similar to the population in all essential characteristics.*" In financial markets, evidence on extrapolation comes from surveys on investors' expectations (Shiller (2000); Dominitz and Manski (2011); Greenwood and Shleifer (2014)) as well as from actual investment decisions (Benartzi (2001); Greenwood and Nagel (2009); Baquero and Verbeek (2008)).

<sup>2</sup>Paul Krugman (2007): "*[Asset backed securities] were promoted as ways to spread risk, making investment safer. What they did instead -aside from making their creators a lot of money [-.]- was to spread confusion, luring investors into taking on more risk than they realized.*" George Soros (2009): "*Securitization was meant to reduce risks [-.]. As it turned out, they increased the risks by transferring ownership of mortgages from bankers who knew their customers to investors who did not.*"

this investor believes that on average assets in  $\alpha$  have expected value  $\tilde{x}$ . No other information is used for assessing the value of pool  $\alpha$ . In particular, investors do not consider how banks may strategically allocate assets into pools,<sup>3</sup> nor do they draw any inference from market prices (we elaborate below on how this valuation method relates to different aspects of bounded rationality as previously considered in the literature).

We further assume that the draws determining the representative samples are made independently across investors. This implies that if assets are sold separately or in homogeneous packages the sampling procedure leads investors to the correct evaluation, and there is no disagreement. By contrast, if the underlying assets of a given package are more heterogeneous, the evaluations of the package are more dispersed across investors, which is consistent with the observation that even highly sophisticated investors exposed to the same objective information may end up with very different assessments of asset-backed securities (Bernardo and Cornell (1997); Carlin, Longstaff and Matoba (2014)).<sup>4</sup>

Creating more dispersion among investors' evaluations may or may not be beneficial to the banks. Some investors tend to overestimate and other investors tend to underestimate the value of the pool (across investors, estimations are on average correct). The question for banks is whether market clearing prices are determined by more or less optimistic investors.

One key determinant is investors' wealth. We assume that investors are risk neutral, wealth-constrained and cannot short sell (an assumption we relax in the extension section). Each investor allocates his whole wealth to the securities perceived as most underpriced. Pooling heterogeneous assets excludes from trading those investors who end up with low valuations, and at the same time it extracts more wealth from those investors who end up with good valuations. The larger the wealth, the more optimistic the marginal investor who determines the market clearing price, which in turns increases the incentive for banks to create disagreement by creating large and heterogeneous pools.

We first highlight this mechanism in a monopolistic setting. We characterize

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<sup>3</sup>Through the choice of how heterogeneous the assets are, the bank affects whether small samples are more likely to be representative of the entire pool. If banks were to pool homogenous assets, one draw would be highly representative of the assets in the pool. If banks instead tend to pool assets of heterogeneous quality (as we show they do) this is no longer the case.

<sup>4</sup>Mark Adelson (S&P chief credit officer): *"It [Complexity] is above the level at which the creation of the methodology can rely solely on mathematical manipulations. Despite the outward simplicity of credit-ratings, the inherent complexity of credit risk in many securitizations means that reasonable professionals starting with the same facts can reasonably reach different conclusions."* Testimony before the Committee on Financial Services, U.S. House of Representatives, September 27, 2007. Quoted in Skreta and Veldkamp (2009).

conditions on investors' wealth under which the monopolistic bank prefers to pool all assets into a single bundle, thereby creating the largest dispersion in investors' evaluations. We also define a threshold on investors' wealth such that when wealth falls short of this threshold, disagreement decreases asset prices, and so selling the loans as separate assets is optimal for the bank, but when investors' wealth exceeds the threshold, the bank prefers to sell its loans with some non-trivial packaging.

A second key determinant of banks' strategies is the degree of market competition. We show that the incentives to pool heterogeneous assets are increased in a market with several banks -we mostly focus on the case in which the number of banks is not too small. A key observation is that, in this case, investors who happen to sample the best asset from some bundles must be indifferent between buying any of those, as otherwise the market would not clear. This implies that the price of each bundle is determined only by the value of its best asset.

Each bank has then the incentive to maximize the most valued asset in a bundle, which can be achieved by pooling all its assets into a single bundle. We show that such full bundling is the only equilibrium when the number of banks is sufficiently large, irrespective of investors' wealth. This should be contrasted with the monopolistic case, in which there is no bundling at low levels of wealth.

We also show that, irrespective of its strategy, a bank is worse off when the other banks offer larger bundles. In this case, a lower fraction of investors is attracted in the market (there is a larger chance of sampling in each bundle an asset whose value is lower than the price) and at the same time the other banks receive a larger fraction of investors' wealth. Thus, bundling creates a negative externality on the other banks.

The externality just highlighted gives rise, for a range of investors' wealth, to a kind of Prisoner's Dilemma -which we call Bundler's Dilemma- defined so that full bundling is the only equilibrium and at the same time it is collectively bad for banks, in the sense that if banks could make a joint decision they would rather choose a finer bundling strategy. We characterize in particular conditions under which banks are induced for strategic reasons to pool all their assets into a big bundle, while they would obtain higher prices if they could jointly decide to sell their loans as separate assets.

## Literature

The heuristic followed by our investors builds on several closely-related aspects of bounded rationality previously discussed in the literature. First, our investors extrapolate from small samples as modelled by Osborne and Rubinstein (1998).

As mentioned above, this can be related to the representativeness heuristic and to the law of small numbers discussed by psychologists (as also noted by Spiegel (2006) in an IO context). Our formalization is most similar to Bianchi and Jehiel (2015), but several other models of extrapolative investors have been developed including De Long, Shleifer, Summers and Waldmann (1990), Barberis, Shleifer and Vishny (1998), Rabin (2002), and Rabin and Vayanos (2010).

Second, investors form their valuations solely based on their sample and not on some prior (knowledge) about the values of the pools. This can be seen as a consequence of the representativeness heuristic (in the form of base rate neglect) or of the fact that investors would perceive the signals received through their samples as much more informative than everything else, as in models with overconfident investors (see for example, Scheinkman and Xiong (2003)).

Third, investors do not realize that other investors too may have private information and that prices may be indicative of the information held by others, which can be related to some versions of overconfidence as well as to the winner's curse. Previous theoretical approaches to the winner's curse include the cursed equilibrium (Eyster and Rabin (2005)) or the analogy-based expectation equilibrium (Jehiel (2005) and Jehiel and Koessler (2008)) that have been applied to financial markets by Eyster and Piccione (2013), Steiner and Stewart (2015), or Kondor and Koszegi (2017).<sup>5</sup>

Compared to the previous behavioral models in financial economics, our focus on the bundling strategies of banks has no counterpart, and as already highlighted it has the effect of structuring the distribution of signals that investors receive, which is not a feature shared by the previous approaches.

Our model is also related to the literature on financial markets with heterogeneous beliefs and short selling constraints as in Harrison and Kreps (1978).<sup>6</sup> A key difference from that literature is that the heterogeneity of beliefs in our setting is not a primitive of the model but it is endogenously determined by the bundling decisions of banks.

Finally, the potential benefits of bundling have been studied in several other streams of literature, from IO to auctions.<sup>7</sup> In particular, a recent literature on obfuscation in IO studies how firms can exploit consumers' naïveté by hiding

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<sup>5</sup>See also Gul, Pesendorfer and Strzalecki (2017) for an alternative modelling of coarseness in financial markets.

<sup>6</sup>See Xiong (2013) for a recent review and Simsek (2013) for a model of financial innovation in such markets

<sup>7</sup>In the context of a monopolist producing multiple goods, see e.g. Adams and Yellen (1976) and McAfee, McMillan and Whinston (1989). For models of auctions, see e.g. Palfrey (1983) and Jehiel, Meyer-Ter-Vehn and Moldovanu (2007).

product attributes or by hindering comparisons across products.<sup>8</sup> Our banks can be viewed as using bundling to make it harder to evaluate their assets, but unlike in models à la Gabaix and Laibson (2006) they cannot make assets more or less visible to investors.

## 2 Model

There are  $N$  risk-neutral banks. Each bank  $i = 1, \dots, N$  possesses several assets. We denote asset  $j$  of bank  $i$  by  $X_j^i$  and its expected payoff by  $x_j^i$ . For concreteness, asset  $X_j^i$  may be thought of as a loan contract with face value normalized to 1, probability of default  $1 - x_j^i \in [0, 1]$ , and zero payoff upon default. We order assets in terms of increasing expected payoff. That is, we have  $x_j^i \leq x_{j+1}^i$  for each  $i$  and  $j$ .

Each bank may pool some of its assets and create securities backed by these pools. Each bank can package its assets into pools as it wishes. We represent the selling strategy of bank  $i$  as a partition of  $X^i = \{X_j^i, j = 1, \dots, J\}$ , denoted by  $\alpha^i = \{\alpha_r^i\}_r$ , in which the set of bundles are indexed by  $r = 1, 2, \dots$ . We focus on complexity considerations that arise merely from banks' bundling strategies. That is, we do not consider the use of possibly complex contracts that would map the value of the underlying pool to the payoff of the securities, and we assume that each bank  $i$  simply creates pass-through securities backed by the pool  $\alpha_r^i$  for each  $r$ . Accordingly, an investor who buys a fraction  $\omega$  of the securities backed by  $\alpha_r^i$  is entitled to a fraction  $\omega$  of the payoffs generated by all the assets in  $\alpha_r^i$ . The expected payoff of bank  $i$  choosing  $\alpha^i$  is defined as

$$\pi^i = \sum_r |\alpha_r^i| p(\alpha_r^i), \quad (1)$$

where  $|\alpha_r^i|$  is the number of assets contained in  $\alpha_r^i$  and  $p(\alpha_r^i)$  is the price of the security backed by  $\alpha_r^i$ . We denote the set of bundles sold by all banks by  $A = \{\{\alpha_r^i\}_r\}_{i=1}^N$ .<sup>9</sup>

There are  $K$  risk-neutral investors, indexed by  $k$ . As investors are risk neutral and they buy claims on the total payoff generated by bundle  $\alpha_r^i$ , they care about the average expected value of the assets in  $\alpha_r^i$ .<sup>10</sup> Having in mind that investors

<sup>8</sup>See Spiegel (2016) for a recent review on these models, and Carlin (2009) for an application of obfuscation to financial products.

<sup>9</sup>For the moment, we take the set of assets which are sold as given and focus on the bundling strategies. We will later introduce the possibility that banks may withhold some of their assets.

<sup>10</sup>The model can be naturally extended, without affecting its main logic, to introduce risk

have no prior knowledge about the value of these assets, we assume that each investor uses the following heuristic procedure. For each bundle  $\alpha_r^i$ , investor  $k$  samples one basic asset from  $\alpha_r^i$  at random (uniformly over all assets in  $\alpha_r^i$ ) and assumes that the average expected value of the assets in  $\alpha_r^i$  coincides with this draw. Specifically, denote by  $\tilde{x}_k(\alpha_r^i)$  the evaluation that investor  $k$  attaches to the average asset in  $\alpha_r^i$ :  $\tilde{x}_k(\alpha_r^i)$  takes value  $x_j^i$  with probability  $1/|\alpha_r^i|$  for every  $X_j^i \in \alpha_r^i$ . This sampling does not vary with the bundling offered by banks, in line with the view that investors do not know ex-ante which assets may be included in the pools, nor do they (need to) know the size of each bundle. We later discuss several modifications of this heuristic procedure.

We assume that the draws are independent across investors.<sup>11</sup> It follows that if  $|\alpha_r^i| = 1$ , investors share the same correct assessment of bundle  $\alpha_r^i$ . But, if  $|\alpha_r^i| > 1$ , investors may attach different values to  $\alpha_r^i$  depending on their draws.

Prices are determined by market clearing, assuming that each investor  $k$  has a fixed budget denoted by  $w_k$  and he cannot borrow nor short sell (an assumption we discuss in Section 5).<sup>12</sup> The supply and demand of the securities backed by  $\alpha_r^i$  are defined as follows. If  $\alpha_r^i$  consists of  $|\alpha_r^i|$  assets, the supply of  $\alpha_r^i$  is

$$S(\alpha_r^i) = |\alpha_r^i|. \quad (2)$$

The demand for  $\alpha_r^i$  is defined as

$$D(\alpha_r^i) = \frac{1}{p(\alpha_r^i)} \sum_k w_k \lambda_k(\alpha_r^i), \quad (3)$$

where  $\lambda_k(\alpha_r^i) \in [0, 1]$  is the fraction of the budget  $w_k$  allocated to bundle  $\alpha_r^i$ . Given the risk-neutrality assumption, each investor allocates his entire budget to the securities perceived as most profitable. That is, we have

$$\lambda_k(\hat{\alpha}) > 0 \text{ iff } \hat{\alpha} \in \arg \max_{\alpha_r^i \in A} \frac{\tilde{x}_k(\alpha_r^i)}{p(\alpha_r^i)} \text{ and } \tilde{x}_k(\hat{\alpha}) - p(\hat{\alpha}) \geq 0,$$

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aversion as well as more complicated security structures, which would induce investors to focus also on other characteristics of the underlying assets.

<sup>11</sup>More generally, the insights developed below would carry over, as long as there is no perfect correlation of the draws across investors.

<sup>12</sup>Investors' wealth is taken as given. An interesting next step would be to endogenize this wealth, possibly as a function of banks' strategies and of market prices, as well as to analyze its dynamics.

and

$$\sum_{\alpha_r^i} \lambda_k(\alpha_r^i) = 1 \text{ if } \max_{\alpha_r^i \in A} (\tilde{x}_k(\alpha_r^i) - p(\alpha_r^i)) > 0.$$

We allow the number of investors  $K$  to be finite or infinite. When  $K$  is finite, the price  $p(\alpha_r^i)$  may vary stochastically as a function of the profile of realizations of the assessments  $\tilde{x}_{k'}(\alpha_r^{i'})$  of bundles  $\alpha_r^{i'}$  by the various investors  $k'$ . When  $K$  is infinite, we have in mind the limiting case as  $K \rightarrow \infty$  of a setting in which, from the law of large numbers, each asset in bundle  $\alpha_r^i$  is sampled by a fraction  $1/|\alpha_r^i|$  of investors. In this case, the randomness of prices is removed, which simplifies the derivation of some results. We denote by  $W$  the aggregate budget across all investors, i.e.,

$$W = \sum_k w_k.$$

The timing is as follows. Banks simultaneously decide their selling strategies so as to maximize the expected payoff as described in (1); investors assess the value of each security according to the above described procedure and form their demand as in (3); a competitive equilibrium emerges, which determines the price for each security so as to clear the markets for all securities.

### 3 Monopoly

We start by analyzing a monopolistic setting with  $N = 1$  (we omit the superscript  $i$  for convenience). This is the simplest setting to highlight some basic insights, in particular the effect of investors' wealth on the incentives for the bank to bundle its assets. Due to the sampling heuristic followed by investors, bundling creates disagreement among investors, and such a disagreement may or may not be beneficial to the bank, depending on the level of wealth in the economy. Investors' wealth determines, along the distribution of beliefs, who the marginal investor fixing the market clearing price is. The larger the wealth, the more likely it is that more optimistic investors are marginal, and so the larger is the incentive for the bank to create heterogeneous bundles.

#### 3.1 Bundling is Optimal only with Disagreement

A preliminary observation is that some disagreement among investors is needed to make bundling profitable to the bank. This can be shown most simply by



considering the case of a single investor,  $K = 1$ .<sup>13</sup> Assume that the bank sells its assets separately (we refer to this case as full separation). Each asset  $X_j$  is correctly perceived as having value  $x_j$ , so the payoff derived by the bank is:

$$\min(\sum x_j, W). \quad (4)$$

Assume by contrast that the bank pools all its assets in a single bundle  $\alpha$  (we refer to this case as full bundling). A generic loan in the bundle is perceived to have value  $x_j$  with probability  $1/|\alpha|$  for each  $j$  and thus the payoff of the bank is:

$$\sum \frac{1}{|\alpha|} \min(|\alpha| x_j, W).$$

Such a payoff cannot strictly exceed the payoff in (4) due to the concavity of  $\min(., w)$  and Jensen's inequality. The argument extends to any other partition, as reported in the following proposition.

**Proposition 1** *Suppose  $K = 1$ . Irrespective of  $W$ , the monopolistic bank prefers full separation.*

### 3.2 Bundling is Optimal with Enough Wealth

Bundling is profitable to the extent that only the investors who overvalue the bundle (as compared with the fundamental value) are willing to buy. The question is whether the wealth possessed by those investors is sufficient to satisfy the corresponding market clearing conditions at such high prices. An immediate observation is that bundling cannot be profitable if the aggregate wealth  $W$  falls short of the fundamental value of the assets which are sold in the market, since selling assets separately exhausts the entire wealth and the payoff from any bundling cannot exceed  $W$  (while it can sometimes fall short of  $W$  due to the possibly pessimistic assessment of the bundle).

More generally, bundling is used in order to extract a larger share of wealth from investors, taking advantage of the possibly optimistic assessments of the values of bundles. Building on this idea, we establish that if there are at least two investors, one sufficiently rich and another not too poor, then full separation is strictly dominated by full bundling.

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<sup>13</sup>The setting is equivalent to one in which investors' draws are perfectly correlated and, in each bundle, the firm assigns uniform probability to which asset gets drawn.

**Proposition 2** *Suppose  $K > 1$  and there exist two investors  $k_1 \neq k_2$  such that  $w_{k_1} > Jx_J$  and  $w_{k_2} > Jx_1$ . Then full bundling strictly dominates full separation.*

To show the above result, recall that by definition  $x_J = \max_j x_j$  and  $x_1 = \min_j x_j$ . The condition  $w_{k_1} > Jx_J$  ensures that full bundling delivers at least the same payoff as full separation. For each  $x_j$ , investor  $k_1$  can pay  $Jx_j$  when sampling an asset with value  $x_j$ . Hence, irrespective of other investors' wealth, the expected payoff to the bank is at least  $\sum_j \frac{1}{J} Jx_j$ . If in addition we have  $w_{k_2} > Jx_1$ , then full bundling strictly dominates full separation. When investor  $k_1$  draws  $x_1$  and investor  $k_2$  draws  $x_j \neq x_1$  (which occurs with strictly positive probability) investor  $k_2$  drives the price of the bundle strictly above  $Jx_1$ . Hence, the expected payoff from full bundling exceeds  $\sum_j x_j$ , that is, the payoff from full separation.

Following the same logic, we show that if investors are sufficiently wealthy, bundling all assets into one package is optimal.

**Proposition 3** *Suppose  $K > 1$  and  $w_k > Jx_J$  for all  $k$ . Then full bundling strictly dominates any other strategy.*

The intuition can be understood as follows (we refer to the Appendix for the complete proof). When investors are very wealthy, the price of any security is determined by the investors with the most optimistic evaluation of the corresponding bundle -that is, by the maximum of the draws across investors- irrespective of the bank's bundling strategy. When a single asset is added to a bundle  $\alpha_r$ , it increases the dispersion in investors' evaluations about  $\alpha_r$ . This increases the payoff of the bank, since the maximal evaluation of the bundle does not decrease and more assets are sold at such maximal evaluation. Hence, it is optimal for the bank to create as much disagreement as possible, which is attained by pooling all assets in a single bundle.

More generally, the larger the aggregate wealth  $W$ , the more profitable it is to create bundles with several assets of heterogeneous value. While full bundling is optimal when  $W$  is large enough, some bundling is optimal at intermediate levels of wealth. (We provide a more detailed analysis along those lines in the Online Appendix.) At low levels of wealth, the bank finds it optimal to sell its assets separately. In particular, if wealth is so low that pooling  $\{X_1, X_2\}$  and offering the other assets separately is dominated by offering all assets separately, then no other bundling can be profitable. This allows us to define a lower bound on wealth such that some bundling is optimal for the bank, as shown more precisely in the next proposition.

**Proposition 4** *Suppose  $K \rightarrow \infty$  and  $J > 1$ . Some bundling dominates full separation if and only if  $W > 2(x_2 + x_1)$ .*

## 4 Oligopoly

We now analyze the effects of competition. We first show that the incentives to offer assets in bundles are larger in a market with several banks. As it turns out, when  $N$  is large, full bundling is the only equilibrium. This is true even at levels of wealth at which a monopolistic bank would sell its assets separately.

We also observe that bundling creates a negative externality on the other banks in the sense that -irrespective of its strategy- each bank is better off when the other banks choose finer partitions than when they choose coarser partitions (in a sense we will make precise). This externality leads to the following possibility: Full bundling can be the only equilibrium and at the same time be detrimental to banks, in the sense that they would be better off if they could coordinate on a different strategy. This is similar to the so-called Prisoner's Dilemma in game theory, and we refer to this as the Bundler's Dilemma.<sup>14</sup>

### 4.1 Full Bundling is the Only Equilibrium

Throughout the section, we assume  $K \rightarrow \infty$  and that all banks have the same set of assets  $X = \{X_j, j = 1, \dots, J\}$  with  $J \geq 2$  and  $0 < x_1 < \dots < x_J$ . Consider some partition of assets across banks. Let  $\alpha_r$  be a generic bundle (the identity of the selling bank is not important),  $J_r \geq 1$  the number of elements in  $\alpha_r$ ,  $x_r^*$  the highest value of the assets in bundle  $\alpha_r$ ,  $p_r$  the market clearing price of a security backed by  $\alpha_r$  and define

$$\mu_r \equiv \frac{p_r}{x_r^*}.$$

We first show that, when  $N$  is large, market clearing requires that the ratio  $\mu_r$  is constant across all bundles sold by all banks.

**Lemma 1** *There exists a  $\mu_0 \in (0, 1]$  and a  $N_0$  such that if  $N \geq N_0$  then market clearing requires*

$$p_r = \mu_0 x_r^* \text{ for all } \alpha_r \in A. \quad (5)$$

*Moreover,  $N_0$  can be chosen irrespective of the partition of assets into bundles.*

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<sup>14</sup>We thank Laura Veldkamp for suggesting this terminology.

The ratio  $\mu_r$  determines the attractiveness of bundle  $r$  for investors who happen to sample the best asset  $x_r^*$ . When  $N$  gets large, the fraction of investors who sample the best asset from at least two bundles gets large and so these investors must be indifferent between any of those bundles. If a bundle had a strictly lower  $\mu_r$ , it would attract a large fraction of wealth and so its price would be large, in contradiction with the assumption that the bundle has the lowest  $\mu_r$ . The markets would not clear unless the ratios  $\mu_r$  are equated across the various bundles. That  $N_0$  can be set independently of the partitions of assets into bundles follows because there are only finitely many possible partitions of the assets of a single bank.

Condition (5) implies that, when  $N$  is large, the price of each bundle is driven by its highest valued asset. This suggests that each bank has the incentive to maximize the most valued asset in a bundle, which can be achieved by pooling all assets in a single bundle. Of course, this loose intuition does not take into account that the constant of proportionality  $\mu_0$  depends itself on the bundling strategies of the banks. But, as it turns out, full bundling is the only equilibrium when  $N$  is large given that in this case a single bank cannot affect much  $\mu_0$ .

**Proposition 5** *Irrespective of  $W$ , there exists a  $N^*$  such that if  $N \geq N^*$  then full bundling is the only equilibrium.*

To have a finer intuition as to why full bundling is an equilibrium, suppose all banks propose the full bundle and bank  $j$  deviates to another partition. From Lemma 1, the fraction of wealth allocated to each bundle depends on the value of its best asset. Full bundling gives a price proportional to  $x_J$  for all assets, while the deviating bank would at best sell  $J - 1$  assets at a price proportional to  $x_J$  and one asset at a price proportional to its second best asset  $x_{J-1}$ . Relative to the other banks, the deviating bank would experience a loss proportional to  $(x_J - x_{J-1})$ , and this remains positive irrespective of  $N$ . At the same time, all banks could benefit from the deviation if the total amount of wealth invested were to increase. Such an increase is bounded by the fraction of wealth not invested before the deviation, which corresponds at most to those investors who sample no best asset from any of the bundles. When  $N$  is large, these investors are not many, and so the increase in wealth is small, which makes the deviation not profitable.

The proposition also rules out any other possibly asymmetric equilibrium. Starting from an arbitrary profile of (possibly asymmetric) bundles, we show that the firm receiving the lowest payoff would be better off by deviating to full bundling.

## 4.2 The Bundler's Dilemma

Another important result that builds on Lemma 1 is that each bank is better off when the other banks choose finer partitions than when they offer coarser partition of their assets. Let us introduce the following definition.

**Definition 1** *Consider two partitions  $\tilde{\alpha}^i$  and  $\alpha^i$  of  $X^i$ . We say that  $\tilde{\alpha}^i$  is coarser than  $\alpha^i$  (or, equivalently, that  $\alpha^i$  is finer than  $\tilde{\alpha}^i$ ) if  $\tilde{\alpha}^i$  can be obtained from the union of some elements of  $\alpha^i$ .*

We can show that, irrespective of its strategy, each bank receives lower payoffs when the other banks offer coarser partitions than when they offer finer partitions. When the other banks offer coarser partitions, the total amount of wealth invested is lower since the probability of sampling an asset whose value is lower than the price from all bundles is larger. At the same time, from Lemma 1, banks offering coarser partitions receive a larger fraction of this wealth as some of their best assets would be included in larger bundles. We then have the following proposition.

**Proposition 6** *Consider partitions  $\tilde{\alpha}$  and  $\alpha$ , where  $\tilde{\alpha}$  is coarser than  $\alpha$ . If  $N \geq N_0$ , irrespective of its strategy, each bank is better off when all other banks offer partition  $\alpha$  than when they offer partition  $\tilde{\alpha}$ .*

Proposition 6 implies in particular that each bank is better off when the other banks sell their asset separately than when they offer them in bundles. In this sense, we say that bundling creates a negative externality on the other banks.

This externality leads to a new phenomenon, which we call Bundler's Dilemma (with obvious reference to the classic Prisoner's Dilemma). Full bundling can be the only equilibrium and at the same time be collectively bad for banks, in the sense that if banks could make a joint decision they would rather choose a finer bundling strategy.

**Definition 2** *We have a Bundler's Dilemma when i) Full bundling is the only equilibrium, and ii) Banks would be better off by jointly choosing a finer bundling strategy.*

A special (extreme) case of the Bundler's Dilemma arises when banks would be collectively better off by selling their assets separately, while in equilibrium they are induced to offer the full bundle. This occurs under the following conditions.

**Corollary 1** *Suppose that  $N \geq N^*$  and*

$$\frac{W}{N} \in (Jx_1, \frac{\sum x_j}{1 - (\frac{1}{J})^N}). \quad (6)$$

*We have a Bundler's Dilemma in which full bundling is the only equilibrium while banks would collectively prefer full separation.*

Equation (6) follows from simple algebra. When  $W/N > Jx_1$ , the price of each bundle is strictly greater than  $x_1$ . Otherwise, all investors would be willing to buy irrespective of their draw, all wealth would be extracted, and the price of each bundle would exceed  $x_1$ , leading to a contradiction. It follows that investors who draw  $X_1$  from all bundles, that is a fraction  $(1/J)^N$  of investors, do not participate and each bundle gets at most

$$\frac{W}{N}(1 - (\frac{1}{J})^N). \quad (7)$$

The upper bound in (6) is derived by imposing that (7) does not exceed  $\sum x_j$  and so each bank would be better off by selling its assets separately.

In a Bundler's Dilemma as defined in (6), the mere option of banks to offer assets in bundles, together with investors' inability to correctly assess the values of the bundles, makes investors better off. There are levels of  $W$  such that prices would be equal to fundamentals if banks offered assets separately and turn out to be below fundamentals only due to bundling. Moreover, as we discuss in the next section, a Bundler's Dilemma may occur even if banks could withhold some of their assets. In this case, a Bundler's Dilemma would prevent banks and investors from fully exploiting potential gains from trade.

## 5 Extensions

Our model while simple is open to many extensions. In this section, we suggest the robustness of our insights in several alternative settings. We refer to the Online Appendix for more formal results as well as for additional extensions.

### Capital Constraints and Short Selling

In our model, the valuation of a given pool across investors is on average correct. The price of the pool however need not reflect such average valuation: optimistic investors cannot invest more than their wealth, pessimistic investors cannot short

sell. Moreover, when facing several bundles, investors only trade those bundles perceived as most profitable. In this way, some beliefs do not influence prices. Capital constraints, and not the absence of short selling per se, are the key reasons why heterogeneous valuations, even if correct on average, may lead to mispricing.

In fact, as we show, introducing short selling may even increase the incentives for banks to create disagreement in a competitive setting. Suppose that short selling too is limited by capital constraints (say due to collateral requirements) and investors allocate their wealth between buying and short selling. While short selling decreases the payoff from bundling (investors with low evaluations can drive the price down), it also decreases the payoff from deviations. With no short selling, deviations from bundling can attract those investors who refrain from investing as they end up with bad evaluations. If these investors could instead short sell, deviations would attract a lower amount of wealth and so become less profitable. We show in a simple example that introducing short selling can make both bundling and a Bundler’s Dilemma more likely to occur.

### **Varying the Sophistication of Investors**

In the baseline model, we assumed that investors could make one draw for each bundle independently of the number of bundles in the market. In some cases, it may be more meaningful to disentangle the number of draws that an investor can make from the total number of bundles put in the market. Assume that investors never buy bundles for which they have no signal, and let  $M$  denote the number of draws each investor can make ( $N$  denotes as before the number of banks). It is readily verified that Proposition 5 can be reformulated as saying for some  $M^*$  and all  $M > M^*$ , there exists  $N^*$  such that for all  $N > N^*$  full bundling is the only equilibrium. The subsequent considerations on the bundler’s dilemma remain valid.<sup>15</sup>

One may also assume that investors would be aware that their assessments of the bundles are imperfect, thereby applying some discount to their evaluations. If the discount is applied uniformly to all bundles (as one would expect in our baseline setting given that investors were viewed as not knowing the size of each bundle), no qualitative property in our analysis would be affected. Assuming investors know the size of the bundles, they could apply a larger discount to larger bundles, and our qualitative insights would remain the same, as long as the extra

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<sup>15</sup>In the Online Appendix, we consider an example with  $N = M = 2$  in which investors can draw more assets from the same bundle, with no replacement. We observe that in this case, full bundling is more likely to occur than in the baseline model

discount applied to large bundles is not too large (see the Online Appendix for an example).

Finally, one may consider a setting in which investors are heterogeneous in their sophistication, possibly with a fraction of them being fully rational (they can be thought of as making infinitely many draws). We would have a Bundler's Dilemma exactly under the same conditions as in Corollary 1 if rational investors are not too many. These investors would still find it optimal to buy the bundle as its price is below the fundamental.<sup>16</sup>

### **When Banks Decide what to Sell**

The previous analysis has focused on banks' bundling strategy taking as given the set of assets sold in the market. Suppose at the other extreme that banks have no fundamental reason to sell the assets (assets have the same value for banks as for investors) and they can decide which assets to sell in addition to their bundling. This modification does not substantially affect our main insights. In a monopolistic setting, whenever a bank chose to sell an asset separately in our baseline model, it would now keep the asset. Selling an asset separately would not give higher payoffs than keeping the asset, and at the same time it would reduce the amount of wealth that investors could use to buy the bundles.

In oligopoly, banks may still be led for strategic reasons into a Bundler's Dilemma. If banks were to coordinate, they would limit the supply of low quality assets at a point in which the benefit of adding an extra asset to their bundles would be offset by the induced reduction in the price of the bundles. A single bank however does not internalize this effect. This leads to an excess supply of low quality assets and so to lower prices than what banks could obtain if they could act cooperatively. We illustrate this idea in a simple example with 3 assets in which in equilibrium banks are induced to offer the full bundle while they would be collectively better off if they committed to keeping one bad asset and selling the rest as a bundle.

It is also clear that, in these more general settings, the Bundler's Dilemma affects not only prices but also which assets are sold in the market, and in particular it may induce banks to keep their assets rather than selling them at low prices. The Bundler's Dilemma may then prevent from fully exploiting possible gains from trade between banks and investors, and so induce welfare losses.

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<sup>16</sup>The share of rational investors has to be small enough to make the deviations from full bundling undesirable.



## Bundling and Uncertainty

In the limit of a very large number of investors and for a given bundling strategy, the belief of the marginal investor is deterministic and so the bank does not face any uncertainty in its payoff. In a setting with a finite number of investors, or in which investors have stochastic wealth, the marginal belief becomes stochastic. A (mean-preserving) spread of this belief may increase or decrease the payoff associated to a given bundling strategy, and this depends on the level of aggregate wealth. When wealth is low, the marginal belief is the lowest evaluation in the population, so there is nothing to lose but possibly something to gain from a spread in beliefs. In this case, randomness increases the incentives to bundle. The opposite occurs when wealth is large. Moreover, in these settings, realizations (of wealth or of evaluations) may be worse than expected and so bundling may be profitable ex-ante but detrimental to the bank ex-post.

## 6 Conclusion

We have studied banks' incentives to package assets into composite pools when investors base their assessments on a limited sample of the assets in the pool. While we have focused on a specific heuristic of investors and a specific financial instrument for banks, we believe our approach can be viewed as representative of a more general theme in which investors use simple valuation models -for example, models that worked well for similar yet more familiar products- and product complexity is endogenous.

Future research could explore the incentives for financial institutions to create complexity when investors use other heuristics as well as to investigate other forms of complexity. The evaluation of some financial products could be difficult not only because of the heterogeneity of the underlying assets (which was our focus) but also because of the complex mapping between the value of the underlying and the payoff to investors.<sup>17</sup> Extending our model so as to allow banks to offer more general securities could offer novel perspectives to the standard security design literature.<sup>18</sup>

Finally, one could investigate several empirical implications of our model. Our

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<sup>17</sup>Think for example at the various tranching structures of mortgage-backed securities or at the various ways in which the returns of a structured financial product can be defined in relation to a benchmark index.

<sup>18</sup>See for example DeMarzo and Duffie (1999) and Biais and Mariotti (2005) for classic contributions and Arora, Barak, Brunnermeier and Ge (2011) for a discussion of complexity in security design.

framework can serve as a building block for a systematic investigation of the incentives to issue asset-backed securities along the business cycle. Our analysis suggests that the heterogeneity of the pools tends to be larger in good times, which is consistent with Downing, Jaffee and Wallace (2009) and Gorton and Metrick (2012) in relation to the 2008 crisis. Our analysis also suggests a positive relation between asset complexity and overpricing, as documented in Henderson and Pearson (2011) and Célérier and Vallée (2017), and a negative relation between breadth of ownership and asset returns, which is consistent with Chen, Hong and Stein (2002). Our analysis also suggests a link between these observations and the dispersion of investors' sentiments (or beliefs), which should be the subject of future tests.

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## 7 Omitted Proofs

### Proof of Proposition 1

Denote with  $\alpha = \{\alpha_r\}_r$  with  $r = 1, 2, \dots, R$  an arbitrary partition of the  $\hat{J}$  assets. We denote with  $x^r$  a generic element of bundle  $\alpha_r$  and with  $y = (x^1, x^2, \dots, x^R)$  a generic vector in which one asset of bundle  $\alpha_1$  is associated with one asset from each of the bundles  $\alpha_2, \alpha_3, \dots, \alpha_R$ . We denote with  $Y$  the set of all possible vectors  $y$ . The payoff from selling assets with partition  $\alpha$  is  $\pi(\alpha) = \sum_{y \in Y} \eta(\alpha) \min(W, \pi_0(y))$ , where  $\eta(\alpha) = \prod_r \frac{1}{|\alpha_r|}$ , and  $\pi_0(y) = \sum_{x^r \in y} |\alpha_r| x^r$ . Notice that by definition  $\sum_{y \in Y} \eta(\alpha) = 1$  and so  $\pi(\alpha) \leq W$ . Notice also that  $\pi(\alpha) \leq \sum_{j \in J} x_j$  since by definition  $\sum_{y \in Y} \eta(\alpha) \pi_0(y) = \sum_{j \in J} x_j$ . Hence,  $\pi(\alpha)$  cannot strictly exceed the payoff from selling assets separately, as defined in (4).

### Proof of Proposition 3

The condition on  $w_k$  ensures that whatever the bundling, the price of  $\alpha$  is the maximum of the draws of the various investors. If  $\alpha$  and  $X$  are separate, the issuer gets  $|\alpha| E[\max_k \tilde{X}_k(\alpha)] + X$ . If  $\alpha$  and  $X$  are bundled then the issuer gets  $(|\alpha| + 1) E[\max_k \tilde{X}_k(\alpha \cup X)]$ . Note that  $|\alpha| E[\max_k \tilde{X}_k(\alpha)] + X$  is the same as  $(|\alpha| + 1) E[\tilde{X}(\alpha)]$  where  $\tilde{X}(\alpha) = \max_k [\frac{|\alpha|}{|\alpha|+1} \tilde{X}_k(\alpha) + \frac{1}{|\alpha|+1} X]$  where  $+$  denotes here the classic addition. When  $\alpha$  and  $X$  are bundled,  $\tilde{X}_k(\alpha \cup X)$  is the lottery  $\frac{|\alpha|}{|\alpha|+1} \tilde{X}_k(\alpha) \oplus \frac{1}{|\alpha|+1} X$ . Thus, it is a mean preserving spread of  $\frac{|\alpha|}{|\alpha|+1} \tilde{X}_k(\alpha) + \frac{1}{|\alpha|+1} X$ . Since for any three independent random variables,  $\tilde{Y}_1, \tilde{Y}'_1, \tilde{Y}_2$  such that  $\tilde{Y}'_1$  is a mean preserving spread of  $\tilde{Y}_1$ , we have that  $E[\max(\tilde{Y}'_1, \tilde{Y}_2)] > E[\max(\tilde{Y}_1, \tilde{Y}_2)]$  (this can be verified noting that  $y_1 \rightarrow E_{y_2}[\max(y_1, y_2)]$  is a convex function of  $y_1$ ), we can conclude that  $(|\alpha| + 1) E[\max_k \tilde{X}_k(\alpha \cup X)] > |\alpha| E[\max_k \tilde{X}_k(\alpha)] + X$  and thus full bundling is optimal.

### Proof of Proposition 4

Suppose  $W > \max(2(x_2 + x_1), \sum_j x_j)$ , full separation gives  $\sum_j x_j$ . Suppose the bank bundles assets  $\{X_1, X_2\}$  and sells the other assets separately. Consider first a candidate equilibrium in which investors who sample  $x_2$  from the bundle are indifferent between trading the single asset  $x_j$  and the bundle. That requires  $2x_2/p_2 = x_j/p_j$  for all  $j > 2$ , where  $p_2$  is the price of the bundle and  $p_j$  is the price of the asset  $x_j$ . In addition, we need that  $p_2 + \sum_{j>2} p_j \leq W$ , so aggregate wealth is enough to buy at prices  $p_2$  and  $p_j$ . The above conditions give  $p_2 \leq 2x_2W/(\sum_{j>2} x_j + 2x_2)$ , and  $p_j \leq x_jW/(\sum_{j>2} x_j + 2x_2)$ . In addition, we need that  $p_2 \leq W/2$  so those investors who have valuation  $x_2$  for the  $(x_2, x_1)$  bundle can indeed drive the price to  $p_2$ . Suppose  $2x_2 < \sum_{j>2} x_j$ , we have  $\frac{2x_2}{\sum_{j>2} x_j + 2x_2} < \frac{W}{2}$

and so  $p_2 = \min(2x_2, \frac{2x_2}{\sum_{j>2} x_j + 2x_2} W)$  and  $p_j = \min(x_j, \frac{x_j}{\sum_{j>2} x_j + 2x_2} W)$  for  $j > 2$ . So the payoff of the bank is

$$\min(W, 2x_2 + \sum_{j>2} x_j),$$

which exceeds  $\sum_j x_j$ . Suppose  $2x_2 \geq \sum_{j>2} x_j$ , which can only occur if  $J = 3$  and  $2x_2 \geq x_3$ . Then we must have  $p_2 = W/2$ , and  $p_3 = x_3 W / 4x_2$ . That cannot be in equilibrium since investors who sample  $x_l$  still have money and would like to drive the price  $p_3$  up. So if  $2x_2 > x_3$  investors are indifferent only if  $p_2 = 2x_2$  and  $p_3 = x_3$ . That requires  $W > 4x_2$ . If  $W < 4x_2$ , then we must have  $p_2 < 2x_2 \frac{p_3}{x_3}$ . If  $W \in (2x_3, 4x_2)$ , we have  $p_2 = \frac{W}{2}$  and  $p_3 = x_3$ . If  $W < 2x_3$ , we have  $p_2 = p_3 = \frac{W}{2}$ . The payoff of the bank is

$$\min(W/2, 2x_2) + \min(W/2, x_3),$$

which also exceeds  $\sum_j x_j$ . Suppose  $W \leq \max(2(x_2 + x_1), \sum_j x_j)$ . If  $W \leq \sum_j x_j$ , then no bundling strictly dominates full separation. If  $W \in (\sum_j x_j, 2(x_2 + x_1)]$ , we must have  $\sum_j x_j < 2(x_2 + x_1)$ , that cannot be for  $J > 3$ . For  $J = 3$  and  $W \leq 2(x_2 + x_1)$ , no bundling strictly dominates full separation.

### Proof of Lemma 1

Denote with  $\mathcal{H}$  the set of (possibly identical) bundles  $r \in \arg \min_r \mu_r$  and with  $\mathcal{L}$  the set of (possibly identical) bundles  $r \notin \arg \min_r \mu_r$ , with  $|\mathcal{H}| = H$  and  $|\mathcal{L}| = L$ . Suppose by contradiction equation (5) is violated, then  $H \geq 1$  and  $L \geq 1$  and

$$\mu_r < \mu_{\tilde{r}} \text{ for all } r \in \mathcal{H} \text{ and all } \tilde{r} \in \mathcal{L}. \quad (8)$$

Given (8), the  $H$  bundles would attract at least all those investors who sample  $x_r^*$  from at least one bundle  $r \in \mathcal{H}$ , and so at least

$$W_{\hat{r}} = (1 - \prod_{r \in \mathcal{H}} (\frac{J_r - 1}{J_r})) W.$$

The other bundles would attract at most the remaining wealth  $W - W_{\hat{r}}$ . Denote with  $\hat{r} \in \mathcal{H}$  the bundle which receives the largest fraction of  $W_{\hat{r}}$ , it would attract at least  $1/H$  of it. Similarly, denote with  $\bar{r} \in \mathcal{L}$  the bundle which receives the lowest fraction of  $W - W_{\hat{r}}$ , it would attract at most  $1/L$  of it. This implies that

$$p_{\hat{r}} \geq \min(x_{\hat{r}}^*, \frac{1}{H} \frac{W_{\hat{r}}}{J_{\hat{r}}}),$$

and

$$p_{\bar{r}} \leq \frac{1}{L} \frac{W - W_{\hat{r}}}{J_{\bar{r}}}.$$

Notice that if

$$x_{\hat{r}}^* \leq \frac{1}{H} \frac{W_{\hat{r}}}{J_{\hat{r}}},$$

then  $\mu_{\hat{r}} = 1$  and so  $\mu_{\hat{r}} < \mu_{\bar{r}}$  would imply  $p_{\bar{r}} > x_{\bar{r}}^*$ , which violates market clearing. Hence,  $\mu_{\hat{r}} < \mu_{\bar{r}}$  requires

$$\frac{1}{H} \frac{W_{\hat{r}}}{J_{\hat{r}}} \frac{1}{x_{\hat{r}}^*} < \frac{1}{L} \frac{W - W_{\hat{r}}}{J_{\bar{r}}} \frac{1}{x_{\bar{r}}^*}, \quad (9)$$

which gives

$$L < \frac{J_{\hat{r}} x_{\hat{r}}^*}{J_{\bar{r}} x_{\bar{r}}^*} \frac{W - W_{\hat{r}}}{W_{\hat{r}}} H. \quad (10)$$

Notice that the r.h.s. of equation (10) decreases in  $H$  and tends to zero as  $H \rightarrow \infty$ . In fact we can write

$$\prod_{r \in \mathcal{H}} \left( \frac{J_r - 1}{J_r} \right) = z^H,$$

for some  $z \in (0, 1)$  and so

$$\frac{W - W_{\hat{r}}}{W_{\hat{r}}} H = \frac{z^H}{1 - z^H} H. \quad (11)$$

We notice that the r.h.s. of equation (11) decreases in  $H$  iff  $H \ln z - z^H + 1 < 0$ , that  $H \ln z - z^H + 1$  decreases in  $H$  and that  $\ln z - z + 1 < 0$  for all  $z \in (0, 1)$ . Hence, condition (10) is violated if either  $H$  or  $L$  (or both) grow sufficiently large, which must be the case when  $N \rightarrow \infty$  since  $H + L \geq N$  (the total number of bundles cannot fall short of the number of banks). Hence, there exists a  $N_0$  such that equation (5) must hold for  $N \geq N_0$ , which proves our result.

### Proof of Proposition 5

We first show that full bundling is an equilibrium when  $N$  is large. We then show that no other bundling can be an equilibrium when  $N$  is large.

**Part 1.** If  $N$  is sufficiently large, full bundling is an equilibrium for all  $W$ .

Suppose all banks offer the full bundling and denote with  $\pi^F$  the payoff for each bank. If  $(1 - (\frac{J-1}{J})^N) \frac{W}{N} \geq Jx_J$  then  $\pi^F = Jx_J$  and no other bundling can increase banks' payoffs. Suppose then

$$\left(1 - \left(\frac{J-1}{J}\right)^N\right) \frac{W}{N} < Jx_J. \quad (12)$$



We have

$$\pi^F \geq \left(1 - \left(\frac{J-1}{J}\right)^N\right) \frac{W}{N}. \quad (13)$$

The condition is derived by noticing that each bundle attracts at least those who sample the maximal asset in the bundle  $x_J$ . These investors are willing to invest all their wealth since the price of the bundle is strictly lower than their evaluation. From (12), the price of the bundle is lower than  $x_J$ .

Suppose bank  $j$  deviates and offers at least two bundles, indexed by  $r$ . The payoff of the deviating bank is  $\pi^j = \sum_{r \geq 1} J_r p_r$ . From (5) we have

$$\pi^j = \mu_0 \sum_{r \geq 1} J_r x_r^*, \quad (14)$$

and so

$$\pi^j \leq \mu_0 (J-1)x_J + \mu_0 x_{J-1}. \quad (15)$$

Notice also that  $\pi^j + (N-1)J\mu_0 x_J \leq W$  and since from (14)  $\pi^j \geq \mu_0 \sum x_j$ , we have

$$\mu_0 \leq \frac{W}{\sum x_j + (N-1)Jx_J}.$$

Together with (15), that gives

$$\pi^j \leq \frac{((J-1)x_J + x_{J-1})W}{\sum x_j + (N-1)Jx_J}.$$

In order to show that the deviation is not profitable, given (13), it is enough to show that

$$\frac{((J-1)x_J + x_{J-1})W}{\sum x_j + (N-1)Jx_J} < \left(1 - \left(\frac{J-1}{J}\right)^N\right) \frac{W}{N}. \quad (16)$$

Equation (16) can be written as

$$\left(\frac{J-1}{J}\right)^N < \frac{\sum x_j - Jx_J + N(x_J - x_{J-1})}{\sum x_j + (N-1)Jx_J}. \quad (17)$$

Notice that the l.h.s. of equation (17) decreases monotonically in  $N$  and tends to zero as  $N \rightarrow \infty$ , while the r.h.s. of equation (17) increases monotonically in  $N$  and tends to

$$\frac{x_J - x_{J-1}}{Jx_J} > 0,$$

as  $N \rightarrow \infty$ . Hence,  $\pi^j < \pi^F$  and so full bundling is an equilibrium for  $N$  sufficiently

large.

**Part 2.** If  $N$  is sufficiently large and irrespective of  $W$ , no alternative bundling is an equilibrium.

Suppose there is one bank, say bank  $j$ , which offers at least two bundles and it deviates by offering the full bundle. From (5), the payoff of the deviating bank can be written as  $\mu_0 J x_J$ . If  $\mu_0 = 1$ , then the deviation is profitable since any other bundling would give strictly less than  $J x_J$ . Suppose then  $\mu_0 < 1$ . As the price of each bundle is strictly lower than the best asset from the bundle, those who sample  $x_r^*$  from at least one bundle  $r$  will invest all their wealth. The total amount of wealth invested is then at least

$$\hat{W} = 1 - \prod_r \left( \frac{J_r - 1}{J_r} \right)^M,$$

where  $M$  is the number of bundles offered after the deviation. Since  $M \geq N$  and  $J_r \leq J$  for all  $r$ , we have

$$\hat{W} \geq 1 - \left( \frac{J - 1}{J} \right)^N. \quad (18)$$

Consider first a candidate symmetric equilibria in which the payoff of the non-deviating banks is the same and it is denoted by  $\pi^{-j}$ . By definition we have  $\pi^j + (N - 1)\pi^{-j} \geq \hat{W}$  and  $\pi^{-j} \leq \mu_0 x_{J-1} + (J - 1)\mu_0 x_J$ , which gives  $\mu_0 J x_J + (N - 1)\mu_0 x_{J-1} + (N - 1)(J - 1)\mu_0 x_J \geq \hat{W}$ . Hence,

$$\mu_0 \geq \frac{\hat{W}}{J x_J + (N - 1)x_{J-1} + (N - 1)(J - 1)x_J},$$

and so

$$\pi^j \geq \frac{\hat{W} J x_J}{J x_J + (N - 1)x_{J-1} + (N - 1)(J - 1)x_J}.$$

Since the payoff before deviation was at most  $W/N$ , the deviation is profitable if

$$\frac{J x_J (1 - (\frac{J-1}{J})^N) W}{J x_J + (N - 1)x_{J-1} + (N - 1)(J - 1)x_J} > \frac{W}{N},$$

which writes

$$x_J - x_{J-1} > \frac{N}{N - 1} \left( \frac{J - 1}{J} \right)^N J x_J,$$

and that shows that  $\pi^j > \pi^F$  and so the alternative bundling is not an equilibrium for for  $N$  sufficiently large.

Suppose there exists an asymmetric equilibrium. Denote with  $\pi^j$  the payoff of a generic bank  $j$  in such equilibrium, we must have  $\min_j \pi^j < \max_j \pi^j$ . Consider

bank  $\tilde{j} \in \arg \min_j \pi^j$ . Suppose bank  $\tilde{j}$  deviates and offers the full bundle. From the above argument, its payoff after deviation would be at least  $\hat{W}/N$  and from (18)  $\hat{W} \rightarrow W$  as  $N \rightarrow \infty$ . Since  $\tilde{j} \in \arg \min_j \pi^j$ , we have  $\pi^{\tilde{j}} < W/N$ . Hence,  $\pi^{\tilde{j}} < \hat{W}/N$  for  $N$  sufficiently large, which rules out the possibility of asymmetric equilibria when  $N$  is large.

### Proof of Proposition 6

Denote with  $r$  the bundles offered by a generic bank  $j$ . Its payoff can be written as  $\pi^j = \sum_{r \geq 1} J_r p_r$  and from (5) we have

$$\pi^j = \mu_0 \sum_{r \geq 1} J_r x_r^*, \quad (19)$$

for some  $\mu_0$ . If all other banks offer a partition  $\alpha = \{\alpha_f\}_f$ , we have

$$\mu_0^F \left( \sum_{r \geq 1} J_r x_r^* + (N-1) \sum_{f \geq 1} J_f x_f^* \right) = W^F, \quad (20)$$

for some  $\mu_0^F$ , and where  $W^F$  is the total amount of wealth invested. Suppose instead that the other banks offer a partition  $\tilde{\alpha} = \{\alpha_c\}_c$ , which is coarser than  $\alpha$ , we have

$$\mu_0^C \left( \sum_{r \geq 1} J_r x_r^* + (N-1) \sum_{c \geq 1} J_c x_c^* \right) = W^C, \quad (21)$$

for some  $\mu_0^C$  and  $W^C$ . Suppose that  $\mu_0^C \geq \mu_0^F$ , then we must have  $W^C \leq W^F$ . The fraction of wealth which is not invested corresponds to the probability that an investor samples an asset with value lower than the price from all bundles. This probability cannot be larger in  $\alpha$  than in  $\tilde{\alpha}$ . By definition, there exists at least one element  $\tilde{\alpha}_c \in \tilde{\alpha}$  which is obtained by the union of at least two elements  $\tilde{\alpha}_f, \hat{\alpha}_f \in \alpha$ . Hence, if  $\mu_0^C \geq \mu_0^F$ , the probability to sample an asset whose value is lower than the price in  $\tilde{\alpha}_c$  cannot be lower than the probability to sample such an asset both in  $\tilde{\alpha}_f$  and in  $\hat{\alpha}_f$ . Notice that from (20) and (21)  $\mu_0^C \geq \mu_0^F$  contradicts  $W^C \leq W^F$  since by definition  $\sum_{c \geq 1} J_c x_c^* > \sum_{f \geq 1} J_f x_f^*$ . Hence, we must have  $\mu_0^C < \mu_0^F$  and from (19) this shows that bank  $j$  receives an higher payoff when the other banks offer a finer partition.

### Proof of Corollary 1

For  $N \geq N^*$ , full bundling is the only equilibrium and it gives a payoff denoted by  $\pi^B$ . For  $W > NJx_1$ , the price of each bundle  $p^B$  is strictly greater than  $x_1$ . If we had  $p^B \leq x_1$  then all investors would be willing to buy irrespective of their draw and for  $p^B \leq x_1$  some investors (say those with valuation  $x_j$  of one bundle) would perceived strictly positive gains from trade. This implies that all wealth  $W$

must be extracted and so  $p^B \geq W/JN$ . For  $W > NJx_1$ , that gives  $p^B > x_1$  which contradicts  $p^B \leq x_1$ .

Since  $p^B > x_1$ , investors who draw  $X_1$  from all bundles will not invest, so

$$\pi^B \leq \frac{1}{N} \left(1 - \left(\frac{1}{J}\right)^N\right) W.$$

If all banks were to offer their assets separately, their payoff would be

$$\pi^S = \min\left(\sum x_j, \frac{W}{N}\right),$$

and we have  $\pi^S > \pi^B$  when

$$\frac{1}{N} \left(1 - \left(\frac{1}{J}\right)^N\right) W < \sum x_j,$$

as required by condition (6).

## 8 Online Appendix

### 8.1 Extensions

#### 8.1.1 Short Selling

Suppose that an investor with wealth  $w$  can short sell  $\theta w/p$  units of an asset of price  $p$  (the baseline model corresponds to  $\theta = 0$ ). Consider a setting with  $N = 2$  banks each with two assets with value  $x_1 = 0$  and  $x_2 > 0$ , and a continuum of investors. Suppose both banks bundle and denote by  $p^B$  the price of the corresponding security. An investor drawing  $x_1$  from one bundle and  $x_2$  from another bundle prefers to buy rather than short selling if

$$p^B \leq \frac{x_2}{1 + \theta}.$$

As we show, bundling can be sustained in equilibrium when  $W$  is sufficiently large, and the required lower bound on wealth is defined by

$$\frac{3 - \theta}{8} \bar{W} = \frac{x_2}{1 + \theta}. \quad (22)$$

In equation (22), the l.h.s. is the payoff from bundling and the r.h.s. is the payoff from deviation. An interesting observation is that both payoffs decrease with  $\theta$ , but the payoff from deviation decreases more. This implies that  $\bar{W}$  decreases in  $\theta$  for  $\theta < 1$ . Hence, bundling is more likely to occur when  $\theta > 0$  than when no short selling is allowed.

A corollary is that short selling creates the possibility of a Bundler's Dilemma, which is not possible with  $N = 2$  and  $\theta = 0$ . The payoff from bundling is lower than what banks would get by jointly deciding not to bundle whenever

$$\frac{3 - \theta}{8} W < x_2.$$

That is, when  $W < \bar{W}(1 + \theta)$ . We then have the following.

**Proposition 7** *Suppose that  $N = 2$ .*

- i) Bundling is an equilibrium iff  $W \geq \bar{W}$ .*
- ii) A Bundler's Dilemma occurs for  $W \in (\bar{W}, \bar{W}(1 + \theta))$ .*

**Proof. Point i).** Suppose all banks bundle. Consider an investor drawing  $x_1$  from first bundle and  $x_2$  from the second bundle. When the price  $p$  of each asset

in each bundle is  $p$ , buying is preferred to short selling if

$$p < \frac{x_2}{1 + \theta}. \quad (23)$$

Suppose that is the case, the price of each asset in each bundle is given by  $\frac{3}{8} \frac{W}{p} = 2 + \frac{1}{8} \frac{\theta w}{p}$ , in which the demand for the bundle comes from investors who have sampled  $x_2$  from at least one bundle (equally shared among the two banks). That gives

$$p = \frac{3 - \theta}{16} W. \quad (24)$$

Conditions (23) and (24) give  $W < 16x_2/(1 + \theta)(3 - \theta)$ . Collecting (23) and (24) we have that the payoff of each bank is  $2p = \min(\frac{3-\theta}{8}W, \frac{2x_2}{1+\theta})$  for  $W < 16x_2/(1 + \theta)(3 - \theta)$ . Suppose now condition (23) is violated. The price of each asset in each bundle is given by  $\frac{1}{8} \frac{W}{p} = 2 + \frac{3}{8} \frac{\theta w}{p}$ , in which the demand for the bundle comes from investors who have sampled  $x_2$  from all bundles (equally shared among the two banks). That gives  $p = (1 - 3\theta)W/16$ , and so the payoff of each bank is  $2p = \min(\frac{1-3\theta}{8}W, 2x_2)$  for  $W \geq 16x_2/(1 + \theta)(1 - 3\theta)$ . That defines  $\pi^\theta$  as

$$\pi^\theta = \begin{cases} \min(\frac{3-\theta}{8}W, \frac{2x_2}{1+\theta}) & \text{for } W \leq \frac{16x_2}{(1+\theta)(1-3\theta)} \\ \min(\frac{1-3\theta}{8}W, 2x_2) & \text{for } W \geq \frac{16x_2}{(1+\theta)(1-3\theta)}. \end{cases} \quad (25)$$

Suppose one bank deviates. If the price of the single asset  $x_2$  is  $p$ , an investor drawing  $x_1$  from the bundle prefers buying the single asset  $x_2$  rather than short selling the bundle if condition (23) holds. In that case, both the bundle and the single asset attract half of the aggregate wealth. The payoff of the deviating bank is  $\min(\frac{1}{2}W, \frac{x_2}{1+\theta})$ . If condition (23) is violated, an investor drawing  $x_1$  from the bundle prefers short selling the bundle rather than buying the single asset  $x_2$ . Only those who sample  $x_2$  from the bundle are willing to buy the single asset  $x_2$ . If  $p_1$  is the price of the single asset  $x_2$  and  $p_2$  is the price of each asset in the bundle, we need

$$p_1 = p_2. \quad (26)$$

Denote with  $\lambda$  the fraction of investors who draw  $x_2$  from the bundle and buy the bundle. For the bundle, we have  $\lambda \frac{W}{2p_2} = 2 + \frac{\theta W}{2p_2}$ . For the single asset, we have  $(1 - \lambda)W = 2p_1$ . Together with (26), these conditions give  $\lambda = (2 + \theta)/3$ , and so  $p_1 = p_2 = W(1 - \theta)/6$ . As we need  $p > x_2/(1 + \theta)$ , we need  $W > 6x_2/((1 - \theta)(1 + \theta))$ . Hence, the payoff of the deviating bank is  $\min(\frac{1-\theta}{6}W, x_2)$  for

$W > 6x_2/((1 - \theta)(1 + \theta))$ . That defines  $\pi^{-\theta}$  as

$$\pi^{-\theta} = \begin{cases} \min(\frac{1}{2}W, \frac{x_2}{1+\theta}) & \text{for } W \leq \frac{6x_2}{(1-\theta)(1+\theta)} \\ \min(\frac{1-\theta}{6}W, x_2) & \text{for } W \geq \frac{6x_2}{(1-\theta)(1+\theta)}. \end{cases} \quad (27)$$

Finally, notice that  $\pi^\theta \geq \pi^{-\theta}$  when  $W \geq \bar{W}$ , where  $\frac{3-\theta}{8}\bar{W} = \frac{x_2}{1+\theta}$ . In fact, the only intersection between  $\pi^\theta$  and  $\pi^{-\theta}$  is at  $\bar{W}$ .

**Point ii).** We need to show that if bundling is an equilibrium then no bundling is not an equilibrium. Suppose no bank bundles and one bank deviates and offers the bundle. Its payoff are defined in the proof defining  $\pi^{-\theta}$  (since we only have 2 banks) and they write as

$$\pi^{-U} = \begin{cases} \min(\frac{1}{2}W, \frac{2x_2}{1+\theta}) & \text{for } W \leq \frac{6x_2}{(1-\theta)(1+\theta)} \\ \min(\frac{1-\theta}{3}W, 2x_2) & \text{for } W > \frac{6x_2}{(1-\theta)(1+\theta)}. \end{cases}$$

We have  $\pi^{-U} > \pi^U$  for  $W > 2x_2$ . Since  $2x_2 < 8x_2/(3 - \theta)(1 + \theta)$ , which is the minimal wealth required to have bundling in equilibrium, we have that if bundling is an equilibrium then no bundling is not an equilibrium. ■

### 8.1.2 Varying the sophistication of investors

Consider the simplest setting with  $J = 2$ ,  $x_1 = 0$  and  $x_2 > 0$ , and  $K \rightarrow \infty$ . Suppose that each investor has a fixed number of signals  $M$  he can draw and assume  $N = M = 2$ . Suppose first that investors sample each bundle at most once (possibly due to the fact that they do not observe the size of each bundle). One can show that our main analysis would not be affected. Suppose instead that investors observe the size of the bundle and that they can sample several assets from the same bundle (they randomly draw across all assets with no replacement). Relative to the baseline model, bundling is now more likely to occur. This is shown in the next proposition (which should be compared to Proposition 7 with  $\theta = 0$ ).

**Proposition 8** *Suppose that  $N = M = 2$ . Bundling is an equilibrium for all  $W$ .*

**Proof.** Suppose  $N = M = 2$ . As there are 4 assets and 2 signals to be drawn, there are 6 possible realizations of the draws, each with equal probability. Suppose all banks bundle. A fraction 1/6 of investors sample two signals from the first bundle, 1/6 of investors sample two signals from the second bundle and the remaining investors sample one signal from each bundle. Hence, for each bundle, 1/3 of investors have valuation  $2x_2$ , 1/6 of investors have valuation  $x_1 + x_2$ , 1/3 of

investors have valuation  $2x_1$  and  $1/6$  of investors have no valuation (and do not buy that bundle).

Each bundle can be sold at price  $2x_2$  when the fraction of investors who draw  $x_2$  from at least one bundle (that is, half of the investors) have enough wealth. That requires  $W/2 > 4x_2$ . When  $W/2 < 2(x_1 + x_2)$ , investors with valuation  $x_1 + x_2$  start buying and so each bundle attracts a fraction  $5/12$  of wealth. The payoff for each bank is

$$\pi_2^B = \begin{cases} \min(\frac{1}{4}W, 2x_2) & \text{for } W \geq 4x_2, \\ \min(\frac{5}{12}W, x_2) & \text{for } W < 4x_2. \end{cases}$$

Suppose bank  $j$  deviates and sell the assets separately, denote its payoff as  $\pi_2^{-B}$ . It can be easily shown that in equilibrium investors who sample  $x_2$  from the bundle and a single asset  $x_2$  always prefer to buy the bundle. Hence, bank  $j$  attracts only those investors who sample  $x_2$  from  $j$  and do not sample  $x_2$  from the bundle (that is,  $1/3$  of investors). We have  $\pi_2^{-B} = \min(W/3, x_2)$ . The result follows from noticing that  $\pi_2^B \geq \pi_2^{-B}$  for all  $W$ . ■

Consider the effect of discounting for complexity. Suppose investors observe the size of the bundle and an investor who samples  $x_j$  from a bundle with two assets has valuation  $x_j - \delta$ , where  $\delta$  is the discount for complexity (evaluations of single assets remain unchanged). We have:

**Proposition 9** *There exists a  $\hat{\delta}$  such that if  $\delta < \hat{\delta}$  then Proposition (5) and Corollary (1) hold. The threshold  $\hat{\delta}$  increases in  $N$  and  $\hat{\delta} \rightarrow x_2$  for  $N \rightarrow \infty$ .*

**Proof.** Suppose all banks bundle and one bank deviates, its payoff is  $\pi^{-D} = \min(x_2, \pi_0^{-D})$ , where

$$\pi_0^{-D} = \max\left(\left(\frac{1}{2}\right)^{N-1}W, \frac{x_2}{(2N-1)x_2 - (N-1)\delta}W\right).$$

Notice that for  $N = 3$ ,  $\pi_0^{-D} = W/4$  iff  $2\delta < h$ . For  $N > 3$ ,  $\pi_0^{-D} = \frac{x_2}{(2N-1)x_2 - (N-1)\delta}W$ . Hence, for  $N = 3$  and  $2\delta < h$ , then bundling is an equilibrium for all  $W$  since  $\frac{1}{3}(1 - (\frac{1}{2})^3)W > \frac{W}{4}$ . When  $\pi_0^{-D} = x_2W/((2N-1)x_2 - (N-1)\delta)$ , bundling is an equilibrium for all  $W$  if

$$\frac{x_2}{(2N-1)x_2 - (N-1)\delta}W < \frac{1}{N}(1 - (\frac{1}{2})^N)W,$$



which defines the upper bound  $\hat{\delta}$ . Notice that for  $N = 3$ ,  $\delta < \hat{\delta}$  implies  $2\delta < h$ . Hence, for  $N > 2$ , bundling is an equilibrium for all  $W$  when  $\delta \leq \hat{\delta}$ . To show the equivalence with Corollary (1), we need to show that if bundling is an equilibrium then no bundling is not an equilibrium. Suppose all banks offer the assets separately and bank  $j$  deviates by offering the bundle  $(x_2, x_1)$ . We can write its payoff as  $\pi^{-U} = \min(2x_2 - \delta, \pi_0^{-U})$ , where

$$\pi_0^{-U} = \min\left(\frac{1}{2}W, \frac{(2x_2 - \delta)}{2x_2 - \delta + (N-1)x_2}W\right). \quad (28)$$

Notice that for  $N \geq 3$ ,  $\pi_0^{-U} = \frac{(2x_2 - \delta)}{2x_2 - \delta + (N-1)x_2}W$  and  $\frac{(2x_2 - \delta)}{2x_2 - \delta + (N-1)x_2} > \frac{1}{N}$ . So for  $N \geq 3$  not bundling is never an equilibrium provided that  $x_2 > \delta$ . ■

### 8.1.3 When banks decide what to sell

The selling strategy of bank  $i$  can be represented as a partition of  $X^i$  which specifies which assets are put in the market and how those assets are bundled. Following the previous notation, such partition can be written as  $\alpha^i = \{\alpha_0^i, \{\alpha_r^i\}_r\}$ , in which the set of assets which are kept in the bank is  $\alpha_0^i$ . The expected payoff of bank  $i$  choosing  $\alpha^i$  is now defined as

$$\pi^i = \sum_r |\alpha_r^i| p(\alpha_r^i) + \sum_{x_j^i \in \alpha_0^i} x_j^i, \quad (29)$$

where  $x_j^i$  is the value of keeping asset  $X_j^i$  in the bank. The simplest setting to illustrate the possibility of a Bundler's Dilemma is one with  $N \geq 3$  banks, each of them with two assets valued  $x_1 = 0$  and one asset valued  $x_2 > 0$ . Full bundling occurs when each bank pools all its assets in a single bundle. We say there is partial bundling when each bank keeps one asset with value  $x_1$  and pools the other assets (with value  $x_1$  and  $x_2$ ) in a single bundle. If all banks offer the full bundle, the payoff of each bank is

$$\pi^F = \min\left(3x_2, \frac{1}{N}\left(1 - \left(\frac{2}{3}\right)^N\right)W\right), \quad (30)$$

where the term  $\frac{1}{N}\left(1 - \left(\frac{2}{3}\right)^N\right)W$  accounts for the total wealth of the share of investors who make a positive draw  $X_2$  for at least one of the  $N$  bundles. Full bundling can be sustained in equilibrium only if  $\pi^F \geq x_2$ , since each bank can get  $x_2$  by

withholding its assets. That requires  $W \geq \tilde{W}$ , where

$$\tilde{W} = \frac{N}{1 - (\frac{2}{3})^N} x_2. \quad (31)$$

As we show in the next proposition full bundling is indeed an equilibrium for  $W \geq \tilde{W}$ . If banks could jointly follow the partial bundling strategy, their payoff would be

$$\pi^P = \min(2x_2, \frac{1}{N}(1 - (\frac{1}{2})^N)W). \quad (32)$$

Full bundling is dominated by partial bundling when  $\pi^F < 2x_2$ , that is for  $W < 2\tilde{W}$ . We can then show the following:

**Proposition 10** *There is a Bundler's Dilemma for  $W \in (\tilde{W}, 2\tilde{W})$ .*

**Proof. Step 1.** We show that full bundling is an equilibrium for  $W \geq \tilde{W}$  when  $N \geq 3$ . To see this, we first show that if bank  $j$  deviates and withdraws one  $x_1$  asset and offers the bundle  $(x_2, x_1)$  its payoff is  $\pi_0^{-F} = x_1 + \min(\pi_0^{-F}, 2x_2)$ , where

$$\pi_0^{-F} = \max(\frac{1}{2}(\frac{2}{3})^{N-1}W, \frac{2}{3N-1}(1 - \frac{1}{2}(\frac{2}{3})^{N-1})W).$$

In  $\pi_0^{-F}$ , the first term corresponds to the case in which the only potential buyers of the deviating bank are those investors who sample  $x_2$  from the deviating bank and  $x_1$  from all the other bundles and the second term corresponds to the case in which investors who sample  $x_2$  from the  $(x_2, x_1)$  bundle and  $x_2$  from at least one  $(x_2, x_1, x_1)$  bundle are indifferent between trading any of those bundles. One can easily show that the first term applies for  $N \leq 3$  and the second term for  $N > 3$ . To see when full bundling is an equilibrium, notice that  $\frac{2}{3N-1}(1 - \frac{1}{2}(\frac{2}{3})^{N-1}) < \frac{1}{N}(1 - (\frac{2}{3})^N)$  for all  $N$ . Notice also that  $\frac{1}{2}(\frac{2}{3})^{N-1} < \frac{1}{N}(1 - (\frac{2}{3})^N)$  for  $N = 3$ .

**Step 2.** We show that full bundling is dominated by partial bundling, that is  $\pi^F < \pi^P$ , when  $W < 2\tilde{W}$ . The payoff  $\pi^F$  is increasing linearly in  $W$  up to  $W^F = 3Nx_2(1 - (\frac{2}{3})^N)^{-1}$ , and it is equal to  $3x_2$  afterwards. The payoff  $\pi^P$  is increasing linearly in  $W$  up to  $W^P = 2Nx_2(1 - (\frac{1}{2})^N)^{-1}$ , and it is equal to  $2x_2$  afterwards. Since  $\pi^F = \pi^P = 0$  at  $W = 0$ ,  $W^F > W^P$ , and  $\frac{d\pi^F}{dW} < \frac{d\pi^P}{dW}$  for  $W < W^P$ , it follows that  $\pi^F < \pi^P$  when  $W < 2\tilde{W}$ .

**Step 3.** We show that if full bundling is an equilibrium then partial bundling is not an equilibrium. We first show that if all banks offer the bundle  $(x_2, x_1)$  and bank  $j$  deviates by offering  $(x_2, x_1, x_1)$  its payoff is  $\pi_0^{-P} = \min(3x_2, \pi_0^{-P})$ , where

$$\pi_0^{-P} = \max(\frac{1}{3}W, \frac{3}{2N+1}(1 - \frac{2}{3}(\frac{1}{2})^{N-1})W). \quad (33)$$

As in  $\pi_0^{-F}$ , the first term corresponds to the case in which investors who sample  $x_2$  from a  $(x_2, x_1)$  bundle and  $x_2$  from the  $(x_2, x_1, x_1)$  bundle prefer buying the latter while the second term corresponds to the case in which these investors are indifferent between the two. To see that if full bundling is an equilibrium then partial bundling is not an equilibrium, notice that  $\frac{3}{2N+1}(1 - \frac{2}{3}(\frac{1}{2})^{N-1}) > \frac{1}{N}(1 - (\frac{1}{2})^N)$  for all  $N$ . Notice that  $\frac{1}{3} > \frac{1}{N}(1 - (\frac{1}{2})^N)$  for  $N = 3$ . Hence, we have  $\pi^{-P} > \pi^P$  for all  $W$  when  $N \geq 3$ . ■

## 8.2 Additional Results in Monopoly

### 8.2.1 Intermediate Wealth

We consider intermediate levels of wealth so as to highlight more generally how investors' wealth affects the incentives to increase the belief dispersion, which in turns determines the optimal form of securitization. We consider the simplest setting for this purpose, one with three assets and a continuum of investors. To illustrate how market clearing prices are set, suppose the bank creates a single bundle consisting of all three assets  $\{X_1, X_2, X_3\}$ . A fraction  $1/3$  of investors samples  $X_1$  and assesses that on average assets in the bundle have value  $x_1$ ;  $1/3$  of investors assesses the average asset as  $x_2$ , and  $1/3$  of investors assesses the average asset as  $x_3$ . If  $W/3$  exceeds  $3x_2$ , the most optimistic investors drive the price to a level at which no other investor is willing to buy. In that case, the payoff of the bank is  $\min(W/3, 3x_3)$ . When  $W/3$  is slightly lower than  $3x_2$ , prices are such that also investors who draw  $x_2$  are willing to buy. Hence, the bank gets  $\min(2W/3, 3x_2)$ . When  $W/3$  is slightly lower than  $3x_1$ , also investors who draw  $x_1$  are willing to buy and the bank gets  $\min(W, 3x_1)$ .

It is clear that when  $W$  is sufficiently large, full bundling dominates any other strategy as it allows to sell assets as if they all had value  $x_3$ . We now characterize more precisely the optimal selling strategy -referred to as partition  $\alpha^*$ - as a function of  $W$ . We show that the larger the aggregate wealth  $W$ , the more profitable it is to create bundles with several assets of heterogeneous value. As  $W$  decreases, the bank prefers to bundle fewer assets and assets of more similar value. When wealth is sufficiently large, full bundling is optimal. The next optimal bundling is  $\{X_1, X_3\}$  followed then by  $\{X_1, X_2\}$  up to the point where it is best to sell assets separately.

**Proposition 11** *Suppose  $K \rightarrow \infty$  and  $J = 3$ . Then*

$$\alpha^* = \begin{cases} \alpha_1 = \{X_1, X_2, X_3\} & \text{for } W \geq 6x_3 + 3x_2 \\ \alpha_1 = \{X_1, X_3\}, \alpha_2 = \{X_2\} & \text{for } W \in [2x_3 + 2x_2, 6x_3 + 3x_2) \\ \alpha_1 = \{X_1, X_2\}, \alpha_2 = \{X_3\} & \text{for } W \in [2x_1 + 2x_2, 2x_3 + 2x_2) \\ \alpha_1 = \{X_1\}, \alpha_2 = \{X_2\}, \alpha_3 = \{X_3\} & \text{for } W < 2x_1 + 2x_2. \end{cases}$$

**Proof.** The payoff from offering the full bundle  $\{X_1, X_2, X_3\}$  is

$$\begin{cases} \min(3x_3, W/3) & \text{for } W \geq 9x_2 \\ \min(3x_2, 2W/3) & \text{for } W \in [9x_2/2, 9x_1/2) \\ \min(3x_1, W) & \text{for } W < 9x_1/2 \end{cases}$$

Suppose instead the bank offers the bundle  $\{X_1, X_2\}$  and  $\{X_3\}$  as separate asset. We show that the payoff for the bank is

$$\begin{cases} \min(W, 2x_2 + x_3) & \text{for } 2x_2 < x_3 \\ \min(W/2, 2x_2) + \min(W/2, x_3) & \text{for } 2x_2 > x_3. \end{cases} \quad (34)$$

In these computations we never consider the possibility that the price of the bundle is driven by its lowest evaluation, since in that case it is clear that bundling cannot strictly dominate full separation. Consider first a candidate equilibrium in which investors who sample  $x_2$  from the  $(x_2, x_1)$  bundle are indifferent between trading the single asset  $x_3$  and the bundle. That requires  $2x_2/p_2 = x_3/p_3$ , where  $p_2$  is the price of the bundle and  $p_3$  is the price of the asset  $x_3$ . In addition, we need that  $p_2 + p_3 \leq W$ , so aggregate wealth is enough to buy prices  $p_2$  and  $p_3$ . The above conditions give  $p_2 \leq 2Wx_2/(x_3 + 2x_2)$  and  $p_3 \leq Wx_3/(x_3 + 2x_2)$ . In addition, we need that  $p_2 \leq W/2$ , so those investors who have valuation  $x_2$  for the  $(x_2, x_1)$  bundle can indeed drive the price to  $p_2$ . Suppose  $\frac{2x_2}{x_3+2x_2} < \frac{1}{2}$  that is  $2x_2 < x_3$ . Then we must have  $p_2 = 2Wx_2/(x_3 + 2x_2)$ , and  $p_3 = Wx_3/(x_3 + 2x_2)$ . So the payoff of the bank is  $\min(W, 2x_2 + x_3)$  when  $2x_2 < x_3$ . Suppose  $2x_2 > x_3$ . Then we must have  $p_2 = W/2$ , and  $p_3 = x_3W/4x_2$ . That cannot be in equilibrium since investors who sample  $x_1$  still have money and would like to drive the price  $p_3$  up. So if  $2x_2 > x_3$  investors are indifferent only if  $p_2 = 2x_2$  and  $p_3 = x_3$ . That requires  $W > 4x_2$ . If  $W < 4x_2$ , then we must have  $p_2 < 2x_2 \frac{p_3}{x_3}$ . If  $W \in (2x_3, 4x_2)$ , we have  $p_2 = \frac{W}{2}$  and  $p_3 = x_3$ . If  $W < 2x_3$ , we have  $p_2 = p_3 = \frac{W}{2}$ .

Consider the other possible partitions. Since  $2x_3 > x_j$  for  $j = 1, 2$ , the payoff follows the second case on the payoff in (34). Hence, for each  $j = 1, 2$ , keeping  $X_j$  and offering the bundle  $\{X_l, X_3\}$  where  $x_j \neq x_l$  gives payoff  $\min(W/2, 2x_3) +$

$\min(W/2, x_j)$ . Comparing the various payoffs, one can see that when  $W < 2(x_1 + x_2)$ , no bundling can strictly dominate full separation. For  $W > 2(x_1 + x_2)$ , the bundling  $\{X_1, X_2\}$  and  $\{X_3\}$  is optimal until  $x_3 + 2x_2 = x_2 + W/2$ , that is for  $W \in [2x_1 + 2x_2, 2x_3 + 2x_2)$ , the bundling  $\{X_1, X_3\}$  and  $\{X_2\}$  is optimal until  $3x_3 + x_2 = W/3$ . For  $W/3 > 3x_3 + x_2$ , the full bundle is optimal. Notice also that the bundling  $\{X_2, X_3\}$  and  $\{X_1\}$  is dominated by  $\{X_1, X_3\}$  and  $\{X_2\}$  for  $W > 2(x_1 + x_2)$ . ■

### 8.2.2 Heterogeneity and Bundle Composition

In the following proposition, we consider the case in which several assets have the same value, and we observe that within homogeneous compositions, it is best to create bundles which are as small as possible.

**Proposition 12** *Suppose  $K \rightarrow \infty$  and the bank has  $2\chi_1$  assets  $Y$  with value 0 and  $2\chi_2$  assets  $Z$  with value  $z > 0$ , where  $\chi_1$  and  $\chi_2$  are positive integers. Then creating a single bundle  $\{2\chi_1 Y, 2\chi_2 Z\}$  is dominated by creating two bundles, each with  $\{\chi_1 Y, \chi_2 Z\}$ .*

**Proof.** The payoff from the single bundle is  $\pi^1 = \min(2(\chi_1 + \chi_2)z, \frac{\chi_2}{\chi_1 + \chi_2}W)$ . The payoff from the two identical bundles is  $\pi^2 = \min(2(\chi_1 + \chi_2)z, (1 - (\frac{\chi_1}{\chi_1 + \chi_2})^2)W)$ . Notice that  $\pi^2 \geq \pi^1$  since  $1 - (\frac{\chi_1}{\chi_1 + \chi_2})^2 > 1 - (\frac{\chi_1}{\chi_1 + \chi_2}) = \frac{\chi_2}{\chi_1 + \chi_2}$ . ■

The intuition behind the result is simple, and can be illustrated when  $\chi_1 = \chi_2 = 1$ . If the monopolist pools all its assets in the bundle  $\{Y, Y, Z, Z\}$ , its payoff is  $\min(4z, W/2)$  since the maximal wealth that can be extracted comes from investors making a  $Z$  draw, i.e., half the population of investors. By creating two bundles  $\{Y, Z\}, \{Y, Z\}$ , its payoff is  $\min(4z, 3W/4)$  where the term  $3W/4$  accounts for the fact that an investor making a good draw from either bundle (there are 3/4 of them) is potentially willing to put his wealth in the market. By disaggregating, the monopolist does not affect the probability of inducing over evaluations of the bundles since the composition of each bundle remains the same. But disaggregating allows the monopolist to extract more wealth since it reduces the fraction of investors who end up with bad draws from all bundles.

### 8.2.3 Risk Aversion

Our main analysis assumes that investors are risk neutral so as to abstract from risk sharing considerations that could motivate bundling. Allowing for risk aversion in our baseline model may actually reinforce the incentives for bundling, as

we now explain. Suppose our basic assets are loans with face value equal to 1 and probability of default equal to  $1 - x_j$ . Suppose defaults are (perceived as) independent across loans. Suppose also that investors observe the size of each bundle.<sup>19</sup> If two assets with respective values  $x_2$  and  $x_1$  are offered separately, the investor perceives an expected value of  $x_2 + x_1$  and a variance of  $x_1(1 - x_1) + x_2(1 - x_2)$ . If the assets are bundled, an investor drawing  $x_2$  believes that the bundle has expected value  $2x_2$  and variance  $2x_2(1 - x_2)$ . If the investor buys  $(x_2 + x_1)/2x_2$  units of the bundle, he perceives the same expected value  $x_2 + x_1$ , but a variance of  $(1 - x_2)(x_2 + x_1)^2/2x_2$ , which is lower than  $x_1(1 - x_1) + x_2(1 - x_2)$ . If investors dislike payoffs with larger variance, the bank has an extra incentive to bundle assets  $x_1$  and  $x_2$ . Bundling may lead investors not only to overestimate the expected value but also to underestimate the variance of returns.

#### 8.2.4 Tranching

An additional motive for bundling assets is to create different tranches which are then sold to investors with different risk appetites. In our setting, tranching can be profitable even if investors are risk neutral. Tranching may be a way to exploit belief heterogeneity and, relative to selling pass-through securities, it may allow the bank to extract a larger share of wealth. We also show this is the case even if the bank were required to keep the most junior tranche.

To see this most simply, consider a bank ( $N = 1$ ) with two assets with  $0 < x_1 < x_2$ . The bank can offer a pass-through security or slice the bundle into a junior and a senior tranche. The senior tranche pays 1 if at least one loan is repaid, the junior tranche pays 1 if both loans are repaid. Assume  $K \rightarrow \infty$ , investors are risk neutral and they observe the size of each bundle. Investors who sample  $x_1$  value the senior tranche as  $s_1 = 1 - (1 - x_1)^2 = 2x_1 - x_1^2$  and the junior tranche as  $j_1 = x_1^2$ . Similarly, investors who sample  $x_2$  value the senior tranche as  $s_2 = 2x_2 - x_2^2$ , and the junior tranche as  $j_2 = x_2^2$ .

We provide some intuition about how the equilibrium works. Suppose the monopolist sells the two tranches and denote its payoff as  $\pi^T$ . At low levels of wealth, everyone buys both tranches and the payoff is  $W$ , which coincides with the payoff from offering a pass-through security  $\pi^B$  when  $W \leq 2x_1$ . When  $W$  is large, those sampling  $x_2$  drive the prices so high that investors sampling  $x_1$  prefer

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<sup>19</sup>This is not needed as the argument would hold irrespective of the size of the bundle, but it simplifies the exposition.

not to buy any tranche. In particular, this occurs when  $W > 2W_4$ , where

$$W_4 = \frac{s_1}{s_2}(j_2 + s_2).$$

In that case, we have again  $\pi^T = \pi^B$ . Tranching is however strictly preferred to the pass-through security for intermediate levels of wealth, for which investors sampling  $x_2$  buy both tranches while those sampling  $x_1$  only buy the senior tranche. Tranching is profitable to the bank as it allows to price discriminate between those having optimistic views and those having pessimistic views about the bundle. This is shown in the next proposition.<sup>20</sup>

**Proposition 13** *We have  $\pi^T \geq \pi^B$  for all  $W$  and  $\pi^T > \pi^B$  for  $W \in (2x_1, 2W_4)$ .*

**Proof.** We first show that the payoff from tranching writes as

$$\pi^T = \begin{cases} \min(\frac{W}{2}, 2(x_1 + x_2)) & \text{for } W \geq 2W_4. \\ \min(W, W_4) & \text{for } W < 2W_4. \end{cases} \quad (35)$$

To see this, notice first that investor drawing signal  $x_z$  for  $z = \{1, 2\}$  prefers to buy the junior tranche as opposed to the senior tranche iff  $p_s/p_j \geq s_z/j_z$ . Notice also that  $s_2/j_2 < s_1/j_1$ , since  $x_1 < x_2$ . This implies that in equilibrium it must be that

$$\frac{p_s}{p_j} = \frac{s_2}{j_2}. \quad (36)$$

Suppose by contradiction that  $p_s/p_j < s_2/j_2$ . Then everyone would strictly prefer buying the senior tranche and no one would buy the junior tranche, which cannot happen in equilibrium. Suppose instead

$$\frac{p_s}{p_j} > \frac{s_2}{j_2}. \quad (37)$$

Those who sample  $x_2$  only buy the junior tranche while those who sample  $x_1$  must buy the senior tranche. Notice first that we must have  $W/2 \leq j_2$ , or those who sample  $x_2$  would still have money left and they would be willing to buy the senior tranche given  $p_s \leq s_1 < s_2$ . Suppose  $W/2 \leq s_1$ . Then those who sample  $x_1$  only buy the senior tranche and  $p_s = W/2$ . We would have  $p_s = p_j = W/2$  and would contradict (37) since  $s_2 > j_2$ . Suppose  $W/2 > s_1$ , which is consistent with

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<sup>20</sup>It should be noted though that  $W_4 < (x_1 + x_2)$ , so that the monopolist stills prefer selling its assets separately rather than as a bundles with two tranches when  $W < 2(x_1 + x_2)$ . Hence, in this example, allowing the monopolist to sell assets in tranches does not strictly improve its payoffs. We suspect it could be otherwise in more elaborate situations.

$W/2 \leq j_2$  only if  $s_1 < j_2$ . We would have  $p_s = s_1$  and  $p_j = W/2$ , which contradicts (37) since  $s_2 > j_2$  and  $W/2 > s_1$ . Hence, it cannot be that those who sample  $x_2$  only buy either the junior tranche or the senior tranche.

Hence, for  $W > 2W_4$ , only those who sample  $x_2$  buy both tranches, in which case  $\pi^T = \min(\frac{W}{2}, 2(x_1 + x_2))$ . If  $W < 2W_4$ , investors sampling  $x_1$  are attracted in the market and they only buy the senior tranche. Investors sampling  $x_2$  buy both tranches and we have  $p_s = s_1$  and  $p_j = \frac{j_2}{s_2} s_1$  due to (36). That gives  $\pi^T = s_1(1 + \frac{j_2}{s_2}) = W_4$ . That occurs until  $W \geq W_4$ . If  $W < W_4$ , we have  $\pi^T = W$ . The payoff from offering the bundle writes as

$$\pi^B = \begin{cases} \min(\frac{W}{2}, 2(x_1 + x_2)) & \text{for } W \geq 4x_1. \\ \min(W, 2x_1) & \text{for } W < 4x_1. \end{cases}$$

Hence, given (35) and noticing that  $W_4 > 2x_1$ , we have  $\pi^T \geq \pi^B$  for all  $W$  and  $\pi^T > \pi^B$  for  $W \in (2x_1, 2W_4)$ . Finally, it should be noted that in our setting the bank would still have an incentive to securitize even if it were required to keep the most junior tranche. Indeed, if investors' wealth is large enough, the bank would still benefit from selling the senior tranche to the most optimistic investors. ■