Bubbles and Crashes with Partially Sophisticated Investors*

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May 29, 2009

Abstract

We analyze bubbles and crashes in a purely speculative market with finite horizon, complete information and in which it is common knowledge that the crash is bound to occur. We assume that some investors are partially sophisticated: While their expectations are consistent with the true average strategies played along the equilibrium, they lack a precise understanding of how these strategies depend on the history of trade. We highlight how euphoria and panic may endogenously arise, define conditions for the existence of equilibrium bubbles and crashes, and investigate whether bubbles may last longer when the amount of fully rational traders increases.

Keywords: Speculative bubbles, crashes, bounded rationality.

JEL codes: D84, G12, C72.

*We thank numerous seminar participants and in particular Cedric Argenton, Abhijit Banerjee, Markus Brunnermeier, Mike Burkart, Sylvain Chassang, Gabrielle Demange, Daniel Dorn, Tore Ellingsen, Guy Laroque, Jacques Olivier, Marco Ottaviani, Peter Raupach, Marcus Salomonsson, Jean-Marc Tallon, Johan Walden, Jörgen Weibull, Muhamet Yildiz and Robert Östling for useful discussions. Milo Bianchi gratefully acknowledges financial support from Region Ile-de-France and from the Knut and Alice Wallenberg Foundation.

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1 Introduction

Speculative bubbles seem widespread (see Garber, 1990; Kindleberger, 2005) but their foundations remain largely unclear. By definition, in speculative bubbles, investors are willing to pay high prices only because they expect the selling price to be even higher in the near future (Stiglitz, 1990). This is typically inconsistent with rational expectations, at least when prices cannot grow without bounds (Tirole, 1982). Speculative bubbles have often been derived in the literature by assuming that some agents are either mechanical in their trading decisions or that they hold exogenously given expectations.[1]

This paper takes a different approach. In our model, the expectations of all agents are endogenously determined and investment strategies are decided optimally given expectations. Agents are neither overconfident nor optimistic. But while some agents are fully rational in the sense of having correct expectations, others are less sophisticated and form their expectations based on some coarse understanding of the working of the market. We are interested in analyzing how bubbles and crashes may endogenously arise within the same framework, and how expectations and investment strategies endogenously evolve along a speculative bubble.

We develop our main ideas in a simple setting intended to be the least conductive to bubbles. We consider a deterministic market with finite horizon and complete information in which agents can trade an asset with no fundamental value. Traders' strategies depend only on their expectations about others' trading strategies, which determine the buy and sell rates and so the asset price in each period.[2]

Importantly, traders are heterogeneous in their ability to understand others' trading strategies, and thus in their expectations about the market dynamics. We consider two extreme types of investors: fully rational investors who have rational expectations, as usual; partially sophisticated investors who understand the aggregate buy and sell rates over the entire duration

[1] More precise references to the literature are provided in Section 1.1.

[2] More precisely, buy and sell rates in every period are respectively defined as the ratios of the effective buy and sell orders to the total amount of possible buyers and sellers in the market.

of the speculative market, without having a precise perception of how these rates vary over the life-cycle of the bubble. Partially sophisticated investors are assumed to adopt the simplest theory of trade volumes and price dynamics that is compatible with their knowledge, thereby expecting constant buy and sell rates throughout the duration of the speculative market, independently from the history of trades.[3] In equilibrium, these constant rates match the aggregate intensities averaged over time, as resulting from the actual buy and sell strategies.

We view the knowledge of the various types of investors as the result of historical learning whereby, when facing a new bubble episode, investors form their optimal strategies based on their knowledge about similar past bubble episodes. Rational investors' knowledge derives from detailed statistics about investment strategies, which allows them to have a precise understanding of the market dynamics. Partially sophisticated investors instead base their knowledge on aggregate statistics, that is, on the buying and selling strategies that apply on average over the bubble.[4]

It is worth emphasizing that in our model all agents whether fully rational or partially sophisticated have enough understanding of the situation to realize that the market is overvalued and bound to crash (indeed, this is common knowledge). Such an understanding is consistent with the evidence that, at least eventually, most agents realize they are in a speculative market, which is documented in Shiller (1989): Shiller reports that just before the U.S. stock market crash of October 1987, 84% of institutional investors thought that the market was overpriced; 78% of them thought that this belief was shared by the rest of investors and, still, 93% of them were net buyers.[5]

[3]This assumption may echo the observation that the day of the crash often appears to be quite similar to many other days. Even the systematic analysis by Cutler, Poterba and Summers (1989) concludes that "many of the largest market movements in recent years have occurred on days when there were no major news events."

[4]For example, these averages may be easier to understand and remember than more detailed information about say the daily buy and sell rates. See Section 2 for further elaboration.

[5]There are also several anecdotes about this. For example, Eric Janszen, a leading commentator of speculative phenomena, wrote in the middle of the Internet bubble (November 1999): "During the final stages, the mania participants finally admit that they are in a mania. But they rationalize that it's OK because they – only they and not the other

Trade is then driven by the fact that all agents believe, rightly or wrongly, that they can profit from investing in the speculative market and exiting at the right time.

The basic mechanism which generates bubbles and crashes in our framework is given by the resulting interaction between investors' historical knowledge and their observation of market trends in the current bubble episode (in line with the experimental evidence in Haruvy, Lahav and Noussair, 2007). Partially sophisticated investors know the average buying and selling strategies averaged along days in which most people want to buy and days in which most people want to sell. As a result, they get a positive surprise after any "good day" and a negative surprise after any "bad day". These surprises characterize our bubble equilibrium, where a series of good days leads to euphoria and one very bad day leads to panic. Specifically, along this equilibrium, investors first observe a series of rising prices, due to excess demand for speculative stocks. Partially sophisticated investors interpret such unexpectedly high prices as a sign that the bubble will last longer.[6] Hence, they decide to remain invested longer than they had planned and, in doing so, they end up overestimating the duration of the bubble. Such investment strategies are exploited by fully rational investors, who feed the bubble for a while and exit just before the (endogenous) crash. Upon observing the massive sale by rational investors, partially sophisticated investors realize it is time to sell (actually, it is too late for most of them), and this indeed leads to the crash.[7]

The rest of the paper is organized as follows. The next subsection reviews some of the key features of our model in relation to the existing literature. In Section 2 we describe the model and the solution concept. In Section 3,

participants – will get out in time." (article accessible at *www.bankrate.com*)

[6] This captures a strong regularity documented in Shiller (2000). As the price increases, more people display "bubble expectations", i.e. the belief that, despite the market being overvalued, it will still increase for a while before the crash.

[7] Kindleberger (2005) provide a rich historical account of such periods of euphoria and panic; Brunnermeier and Nagel (2004) and Temin and Voth (2004) document how in various bubble episodes major investors earned large profits by timing the market correctly; Greenwood and Nagel (2008) show that inexperienced investors sustained the Internet bubble.

we analyze bubble equilibria, deriving their structure in terms of investment strategies and expectation dynamics, characterizing the conditions for their existence, and showing how the maximal duration of bubbles varies as a function of our parameters. In Section 4, inspired by the efficient market hypothesis, we explore the relation between bubbles and the share of rational investors in our setting. We observe that rational investors should be neither too many nor too few for bubble equilibria to arise with the property that, just before the crash, there is a panic phase in which investors realize everyone is trying to sell and the crash is about to occur.[8] We also observe in our basic model that, when there are more rational investors, the maximal duration of a bubble gets smaller. However, by considering a setting with uncertainty aversion, we show that bubbles may last longer as the fraction of rational investors increases.[9] Thus, in our model, whether rational investors have a stabilizing role depends on the attitude of investors toward uncertainty. We conclude in Section 5 by discussing some avenues for extensions. Omitted proofs are provided in the Appendix.

1.1 Related literature

There is a vast literature on speculative bubbles, and we only review some general themes here.[10] Part of the literature builds on the fact that some information, e.g. about the value of fundamentals, is dispersed among agents and that agents hold subjective priors.[11] We differ from this literature first as

[8] With too many rational agents, bubbles cannot arise because we are too close to a rational expectation model. With too few rational agents, partially sophisticated traders do not observe any massive sale before the crash occurs and as a result they do not realize that the crash is about to occur before it actually does.

[9] In fact, facing less strategic uncertainty, rational agents are more prone to invest than partially sophisticated ones. As the share of rational investors increases, more people enter the speculative market, which may induce partially sophisticated agents to be more optimistic, thereby sustaining longer bubbles.

[10] For more detailed reviews, see Bianchi (2007) and Brunnermeier (2007).

[11] From a theoretical viewpoint, private information alone cannot explain bubbles, as can be inferred from the no-trade theorems (Milgrom and Stokey, 1982). Somewhat more generally, Allen, Morris and Postlewaite (1993) show that bubbles may arise in a finite setting with private information only if one also introduces ex-ante inefficiency, short sale constraints, or lack of common knowledge of agents' trades. See, in particular, Morris (1996) and Biais and Bossaerts (1998) for models of speculation with subjective priors.

we consider a setting with complete information. There is substantial experimental evidence that bubbles emerge even in contexts in which the structure of the game and the value of future dividends are, by design, commonly known to subjects (see Porter and Smith, 2003, for a review). Moreover, in our approach, I-agents' expectations are not derived from an exogenous subjective belief but from the coarse (yet correct) perception of others' trading strategies.

Another stream of literature focuses on the effects of purely mechanical traders (De Long, Shleifer, Summers and Waldmann, 1990a) or of agents who form their expectations about future prices simply by extrapolating from past market trends (Cutler, Poterba and Summers, 1990 and De Long, Shleifer, Summers and Waldmann, 1990b). In a phase of rising prices, we note that such agents would expect the prices to increase with no bounds, and so they would never understand that they are in a bubble nor that the market may crash. As emphasized above, we focus instead on agents with enough sophistication to understand that they are in a bubble and that the market must crash.

Moreover, in contrast with our approach, the literature has typically modeled bubbles and crashes separately. For example, Gennotte and Leland (1990) focus on the role of hedge funds in provoking the crash while taking as given the fact that the market is overvalued. Abreu and Brunnermeier (2003) focus on how coordination issues among rational arbitrageurs may delay the crash, while abstracting from the underlying process generating the bubble. On the other hand, De Long, Shleifer, Summers and Waldmann (1990b) explain how feedback trading can generate a bubble, but exogenously impose an end period at which the crash occurs. Scheinkman and Xiong (2003) show how overconfidence can sustain speculative trade but do not consider how the crash may endogenously occur.

Finally, by emphasizing the role of cognitive heterogeneity, our work is related to a wide literature on the limits to information processing.[12] In par-

[12] See in particular Higgins (1996) for an exposition of the idea of accessibility in psychology and Kahneman (2003) for economic applications. Many authors have explored such themes in strategic interactions (see Rubinstein, 1998 and the references therein and Jehiel, 1995; Jehiel, 2005; Jehiel and Samet, 2007), and financial markets (see Hirshleifer,

ticular, our idea of equilibrium is in the spirit of Jehiel (2005) who assumes, in the context of extensive form games, that each player understands only the aggregate behavior of his opponents over a bundle of nodes/states.[13] Our model can be viewed as providing a finance theoretical analog of such concepts and it allows us to shed new light on the emergence of bubbles and crashes and on the stabilizing role of arbitrageurs.

2 The model

Our economy is populated by a unit mass of risk neutral individuals.[14] Initially, a mass K of individuals is endowed with w units of cash and a mass $(1-K)$ is endowed with one unit of stock. The value of cash is constant over time. The stock pays no dividends, its fundamental value is zero.

In each period $t = 1, 2, ...$, individuals can trade. Within each period, individuals simultaneously submit their orders, the stock price p_t is announced, and orders are cleared. For simplicity, we assume that each agent can hold at most one stock at a time, and each stock is indivisible.[15] We also rule out borrowing of stocks or cash. Hence, the investment option for individual i in period t is simply $\{$*buy*, *stay out*$\}$ if i holds cash at t, or $\{$*sell*, *stay in*$\}$ if i holds a stock at t. Each agent chooses the investment strategy which maximizes his expected payoffs, as described below. We first specify the stock price dynamics, as a function of buy and sell orders, and then describe payoffs and expectations.

2.1 Stock price dynamics

While the amount of stocks is fixed to $(1-K)$ throughout the analysis, the amount of people who can buy stocks decreases over time due to the exit of some investors (as explained below). We denote the amount of potential

2001).

[13] A related idea is developed in static games of incomplete information by Eyster and Rabin (2005).

[14] Section 4.3 considers the case of uncertainty averse agents.

[15] The substance of our analysis would not change if stocks were perfectly divisible and everyone could spend his entire wealth in stocks.

buyers as K_t, which can be interpreted as a measure of the aggregate investment capacity in the speculative market at time t. The amount of buy and sell orders at t, denoted respectively by B_t and S_t, can then be written as

$$B_t = \beta_t K_t \text{ and } S_t = \sigma_t(1 - K), \tag{1}$$

where β_t is the share of those potential buyers who effectively want to buy and σ_t is the share of those potential sellers who effectively want to sell in period t.[16]

The stock price p_t is determined in every period t according to the rule

$$p_t = f(N_t, p_{t-1}),$$

where the function $f : [K - 1, K] \times [0, w] \to [0, w]$ is strictly increasing in $N_t \equiv B_t - S_t$. Moreover, we set the initial price of the stock to its fundamental value, i.e. $p_0 = 0$, and we assume that if the amount of buy orders equals the amount of sell orders the price stays the same, i.e. $f(0, p_{t-1}) = p_{t-1}$. This implies that

$$p_t \geq p_{t-1} \Leftrightarrow B_t \geq S_t. \tag{2}$$

While we do not derive the function $f(.)$ from more primitive parameters, note that (2) does not affect the conclusion that no trade would occur in our setting if everyone had rational expectations, which follows because there are no gains from trade in our economy. Moreover, this function can be seen as the reduced form of more standard market clearing mechanisms. For example, when agents submit limit orders, (2) may result from the heterogeneity of the limit prices. To illustrate this mechanism in the simplest way, assume that, when he submits a buy order, an individual specifies a limit price p_t^b. Similarly, in the event of a sell order, he specifies a limit price p_t^s. Let these limit prices be exogenously given for each individual by $p_t^s = \lambda p_{t-1}$ and

[16] These rates are, in principle, the realization of a random variable that aggregates the strategies of every agent at a given point in time. However, since we consider a setting with a continuum of agents, each realization of this variable corresponds to its expected value. Accordingly, in what follows, we simplify the notation and ignore the distinction between the expected values of these quantities in a given period and their actual realizations.

$p_t^b = \lambda p_{t-1} + (1-\lambda)w$, where the parameter λ is drawn independently across individuals from a commonly known distribution with smooth density and support on $[0,1]$.[17] The induced distributions of limit prices p_t^s and p_t^b are described by the cumulative functions Γ_t and Λ_t, respectively. These distributions depend on the history of trades, but from the above assumption their support always lies respectively in $[0, p_{t-1}]$ and in $[p_{t-1}, w]$. It is then clear that if $B_t \geq S_t$, the market clearing price p_t solves $B_t[1-\Lambda_t(p_t)] = S_t$, which implies $\Lambda_t(p_t) \geq 0$ and so $p_t \geq p_{t-1}$. If instead $B_t < S_t$, the market clearing price p_t solves $B_t = S_t[1-\Gamma_t(p_t)]$, which implies $\Gamma_t(p_t) > 0$ and so $p_t < p_{t-1}$. Hence, market clearing would here induce the relation expressed in (2).[18]

Finally, we assume that, at the end of each period t, each agent observes the trading price p_t and the volume of trade $V_t = \min\{B_t, S_t\}$, from which he can correctly infer B_t and S_t. In what follows, we refer to B_t and S_t simply as demand and supply in period t.

2.2 Payoffs and exit from the market

Investors may exit from the speculative market in any period t either by selling or by deciding not to buy at t. In our equilibrium, agents who exit the speculative market at t never decide to re-enter later on (see Lemma 1 below) and this is rightly understood by everybody.[19] Thus, with the notation introduced in 2.1, the amount of exit in period t is defined as

$$E_t = V_t + (1-\beta_t)K_t. \tag{3}$$

The decision to sell at period t may either be deliberate or it may be induced by a liquidity shock. Formally, agents can be in two states. In the normal state, the payoff of an agent is as expected: it is zero if he holds cash or stock

[17] A fully specified model may derive such λ from heterogeneous preferences, concerning for example attitudes towards uncertainty.

[18] With this mechanism in mind, the function f may depend on the history of trade, but as just mentioned, this does not affect our analysis (which relies only on the equivalence (2)).

[19] In Bianchi and Jehiel (2008), we show that, under a (natural) assumption, this is the only consistent case.

forever; $(p_s - p_t)$ if he buys a stock at time t and he sells it at time s; p_s if he initially owns a stock and sells it at s, and $-p_t$ if he buys a stock at t and keeps it forever. With probability $z > 0$, an agent in the normal state who owns a stock may be hit by a liquidity shock. In this case, he only cares about immediate cash and he places no value on cash in the future. He is then induced to sell immediately and to stay out of the market from then on.[20] We assume that the probability of a shock is small so that most agents are in the normal state and $z(1-K) < K$. Liquidity sellers are here introduced in order to impose finite horizon, so that the crash always occurs at the latest at $1 + K/z(1-K)$. Notice however that these agents do not create any gain from trade in our setting (as opposed to standard noise traders). Moreover, we will see that in equilibrium the crash will always occur earlier than at date $1 + K/z(1-K)$.

Given that exits are permanent in equilibrium, it follows that the mass of people K_t who can possibly buy a stock at t evolves as

$$K_{t+1} = K_t - E_t, \tag{4}$$

and, by equation (3), we have

$$K_{t+1} = \beta_t K_t - V_t. \tag{5}$$

From equation (5), it follows that the price p_t never recovers after having dropped. If in period t the price drops, it must be due to excess supply in t, in which case the volume of trade V_t is equal to the demand $\beta_t K_t$ and equation (5) yields $K_{t+1} = 0$. By equation (4), K_t can only decrease over time, which implies that $K_{t+s} = 0$ for all $s \geq 1$. Thus, after a price drop, the market closes. This simplifying feature of our equilibrium allows us to focus on the innovative part of our approach, which is the modeling of agents' expectations about the buy and sell rates.

[20] Assuming, perhaps more naturally, that everyone -not only stock holders- may be hit by a liquidity shock would not change our results, but it would complicate the algebra.

2.3 Cognitive abilities and equilibrium

A key ingredient of our model is that agents differ in their ability to understand other agents' trading strategies, and so in their expectations about the dynamics of trade volumes and the associated prices. Such dynamics depend on demand and supply in each period, which in turn depend on the amount of agents still active in the market together with their buy and sell strategies.

For an agent of type θ, the period s expectations about the demand and supply in period t are given by

$$B_t^{\theta,s} = \beta_t^{\theta,s} K_t^{\theta,s} \text{ and } S_t^{\theta,s} = \sigma_t^{\theta,s}(1-K) \text{ for every } s \leq t, \tag{6}$$

where $\beta_t^{\theta,s}$ and $\sigma_t^{\theta,s}$ are this agent's expected buy and sell rates, and $K_t^{\theta,s}$ is the expected amount of potential buyers at t (see definition (1)). In order to estimate the latter, agents need to know how many traders are in the market at s, and how many traders exit from s to $t-1$. Recall that, in period s, agents have observed the history of prices and trade volumes, from which they can correctly infer the amount of exits until $s-1$ and so K_s. Hence,

$$K_s^{\theta,s} = K_s \text{ for every } \theta \text{ and } s.$$

For $t > s$, using equation (4), we have

$$K_t^{\theta,s} = K_s - \sum_{w=s}^{t-1} E_w^{\theta,s}, \tag{7}$$

where, by equation (3),

$$E_w^{\theta,s} = \min(B_w^{\theta,s}, S_w^{\theta,s}) + (1 - \beta_w^{\theta,s}) K_w^{\theta,s}. \tag{8}$$

Given equations (6), (7) and (8), an agent's expectation about future market dynamics is completely characterized by his expectation about future buy and sell rates. Such expectation depends on the agent's type, as we now describe. For simplicity, we consider a setting with only two cognitive types: standard rational agents R and partially sophisticated agents I, in proportion

r and $(1-r)$, respectively.

Standard rational agents understand perfectly well the patterns of other investors' strategies. Hence, if the actual buy and sell rates arising in equilibrium in period t are given by β_t and σ_t, R-agents' expectations must satisfy

$$\beta_t^{R,s} = \beta_t \text{ and } \sigma_t^{R,s} = \sigma_t \text{ for every } s \leq t. \tag{9}$$

Partially sophisticated agents, on the other hand, expect constant buy and sell rates throughout the duration of the speculative market, where these rates coincide with the actual aggregate intensities averaged over time. Formally, we denote by $T+1$ the last date in which the speculative market operates, as determined endogenously in equilibrium. The average buy rate $\bar{\beta}$ and the average sell rate $\bar{\sigma}$ for the sequence of buy and sell decisions arising in equilibrium are

$$\bar{\beta} = \frac{1}{T+1}\sum_{t=1}^{T+1}\beta_t \text{ and } \bar{\sigma} = \frac{1}{T+1}\sum_{t=1}^{T+1}\sigma_t. \tag{10}$$

I-agents' expectations are required to correspond to such averages, hence we have

$$\beta_t^{I,s} = \bar{\beta} \text{ and } \sigma_t^{I,s} = \bar{\sigma} \text{ for every } s \leq t. \tag{11}$$

After each history of prices and trade volumes, each agent chooses an investment strategy that maximizes his expected payoff, as described above. An investment strategy profile specifies an investment strategy for every agent in the economy, which serves to define an equilibrium in our setting.

Definition 1 *(**Equilibrium**): An investment strategy profile is an equilibrium if, all along the equilibrium path, each agent's investment strategy maximizes his expected payoff, given the expectations defined in equations (9) and (11).*

Remarks on the interpretation of equilibrium:

As mentioned in the Introduction, we think of this equilibrium as the result of a process of learning at an historical level, in which new cohorts of investors enter the market in each bubble episode. Investors interpret the current market in light of historical data about similar episodes. Fully rational investors analyze such data with elaborate statistics, which leads them to know β_t and σ_t and so to rightly understand the trade dynamics. Partially sophisticated investors, instead, use a simplified model, able to provide the correct averages $\bar{\beta}$ and $\bar{\sigma}$ but no more detailed statistics. In a sense, they apply a linear model to analyze trade dynamics that are not necessarily linear (as in the spirit of Sargent, 1993).

It follows that I-investors should not be thought of as deriving their expectations by computing the averages $\bar{\beta}$ and $\bar{\sigma}$. In fact, they do not know the realizations of β_t and σ_t, nor do they know T. They know the average buying and selling strategies, and only these, as the result of extracting aggregate statistics from past data. Hence, there is no way in which, based on their historical knowledge, these investors could update their expectations about the future buying and selling strategies as a function of the history of trade.[21]

Relatedly, even when observing a realization different from the mean, I-agents do not change their model about the underlying distribution of strategies. One can think of several reasons for this. First, they may perceive that some stochastic element, specific to the current bubble, affects both demand and supply. Despite the deterministic character of our model, such a perception would make any realization of prices and trade volumes compatible with the expected $\bar{\beta}$ and $\bar{\sigma}$ (see Section 5.1 for further discussion). Alternatively, they may understand that they only know an average from which actual realizations may differ. Some implications of such understanding are analyzed in Section 4.3. Third, assuming agents adjust their theory when it is proven sufficiently wrong, agents in our model may get a strong enough signal that their trading decisions are wrong only when it is too late. The next section shows a sense in which this is indeed the case in our model.

[21] In particular, as these strategies are all investors know, they need not know that there are different types of investors in the market. Hence, I-agents are not aware that other investors may have a more accurate understanding of the market dynamics. Otherwise, given that in our model trades mostly occur for speculative reasons, I-agents may simply decide to stay outside the market if they realize they are less sophisticated than others.

Finally, it should be mentioned that our definition is reminiscent of the rational expectation equilibrium in that, due to the dynamic nature of the interaction, beliefs and investment strategies must be optimally adjusted at every point in time and beliefs are closed through some consistency criteria. At the same time, since each individual agent has a negligible weight (there is a continuum of agents), this notion is in the spirit of the Nash equilibrium, where no single agent can on his own move the system away from the equilibrium path.[22]

3 Analysis

3.1 Optimal investment strategies

We focus on symmetric equilibria in pure strategies, where all investors of a given type and with a given endowment in period t follow the same pure strategy. Observe first that the existence of an equilibrium is not an issue, as there is always the non-bubble equilibrium in which every agent exits the speculative market at the very first period.[23] Our interest lies in showing the possibility of bubble equilibria, and characterizing the conditions for such equilibria to exist. Given that the fundamental value of the asset is zero, we define any situation in which trade occurs as a bubble. Conversely, if at some point no one is willing to buy the stock at any price, the speculative market closes. Provided that some trade had occurred, we then say that there is a crash.

The problem faced by an individual of a given type is the same irrespective of whether he has cash or stock. That is, for any agent $i \in \theta$ with cash and any agent $j \in \theta$ with stock (who is not hit by a liquidity shock), i prefers to

[22] Indeed, our definition only considers the incentives of agents on the equilibrium path, and not the adjustment of beliefs and strategies after a positive mass of agents have made non-equilibrium decisions. While one could easily amend the solution concept to cover off-the-path optimizations and expectations, this would make the notation heavier (in particular, the state variable parameterizing the decisions should no longer be the calendar time t but the entire history of buy/sell decisions) without adding much economic insight.

[23] In this equilibrium, I-agents' expectation is correct, and their decisions to exit immediately is thus rational.

buy if and only if j prefers to stay in, and i prefers to stay out if and only if j prefers to sell. Intuitively, trade occurs either among people with different needs, as described by the liquidity sellers z, or among those with different expectations, as described by the different types.

In principle, investment strategies may be very complicated, since each agent may condition his current strategy on the whole history of past trades and on his own past trading decisions. However, as it turns out, our model allows a very simple representation of optimal trading strategies. We start by showing that, expecting all exits from the market to be permanent, no agent wishes to re-enter after having exited, thereby justifying the considerations developed in Subsections 2.2 and 2.3.

Lemma 1 *Expecting exits to be permanent, an agent who exits the speculative market at t prefers to stay out from then on.*

While Lemma 1 is proven in the Appendix, the intuition is that if an agent (whether of type I or R) voluntarily decides to exit at t, he must expect the price to drop at $t+1$ and he cannot expect the price to recover later on. This makes subsequent re-entry suboptimal. This observation also implies that optimal trading strategies at time t can be expressed as a function of the expected prices at t and at $t+1$ only. We state the result in the next Proposition.

Proposition 1 *An agent $i \in \theta$ prefers to buy at t if and only if*

$$p_{t+1}^{\theta,t} \geq p_t^{\theta,t}, \tag{12}$$

or, equivalently, if and only if

$$B_{t+1}^{\theta,t} \geq S_{t+1}^{\theta,t}. \tag{13}$$

Proposition 1 allows us to write the optimal investment strategy for I-agents in any period t simply as a function of the observed amount of people who can buy at t and of the *constant* expectation about future buy and sell rates. We express this more precisely in the next Corollary.

Corollary 1 *An agent $i \in I$ prefers to buy at t if and only if*

$$K_t \geq W, \tag{14}$$

where

$$W \equiv \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2}. \tag{15}$$

In sum, at any t, optimal trading strategies for I-agents are only a function of K_t, which describes the history of trades, and of the expectation about future buy and sell rates, as expressed in equation (15). Such expectation, in turn, depends on the equilibrium duration of the bubble, but it remains constant over time. R-agents' expectation, instead, reflects the true strategies observed along the equilibrium, and so it may vary with t. In particular, given (13), these agents buy in period t if and only if $\beta_{t+1}K_{t+1} \geq \sigma_{t+1}(1-K)$.

3.2 Bubble equilibria

We can now show that, under conditions to be characterized in the next subsection, there exist equilibria of the following form. Apart from a share z of stock-holders who sell in each period due to liquidity shocks, investors' strategies are such that in each period $t \leq T-1$ everyone tries to buy and no one wants to sell; in period T, I-investors buy and R-investors sell; at $T+1$, everyone tries to sell but no one is willing to buy. The crash then occurs and the market closes. Along these equilibria, called bubble equilibria, the aggregate buy and sell rates are

$$\beta_t = \begin{cases} 1 & \text{for } t \leq T-1, \\ 1-r & \text{for } t = T, \\ 0 & \text{for } t = T+1, \end{cases} \tag{16}$$

and

$$\sigma_t = \begin{cases} z & \text{for } t \leq T-1, \\ z + r(1-z) & \text{for } t = T, \\ 1 & \text{for } t = T+1. \end{cases} \tag{17}$$

It is not difficult to see that all equilibria in which some trade occurs take the form described by equations (16) and (17). We show this in the following Lemma.

Lemma 2 *All bubble equilibria are described by equations (16) and (17).*

Given the expressions of β_t and σ_t in (16) and (17), and the consistency condition (10), I-agents are induced to expect the following buy and sell rates:

$$\bar{\beta} = \frac{T-r}{T+1}, \tag{18}$$

and

$$\bar{\sigma} = \frac{Tz + r(1-z) + 1}{T+1}. \tag{19}$$

This allows us to define $B_s^{\theta,t}$ and $S_s^{\theta,t}$ according to (6), (7), and (8). Besides, given the above specifications, the only variable remaining to endogenize is the duration T of the bubble. A major objective of the next analysis is then to characterize the conditions for the existence of such T, and to understand how T depends on our exogenous parameters K, z and r.

For the above trading strategies to define an equilibrium, three conditions are required. First, each agent $i \in R$ has to prefer to buy at $T-1$, so we must have $p_T > p_{T-1}$. From equation (13), this condition can be written

$$B_T \geq S_T. \tag{20}$$

Second, each agent $i \in I$ has to prefer to buy at T, which, from equation (14), can be written

$$K_T \geq W. \tag{21}$$

Third, each agent $i \in I$ has to prefer to sell at $T+1$, which, again using equation (14), can be written

$$K_{T+1} < W. \tag{22}$$

The last two conditions also imply that, given I-agents' behavior, each agent $i \in R$ prefers to sell at T since the market crashes at $T+1$. We summarize these observations in the following Proposition.

Proposition 2 *Consider the trading strategies defined by equations (16) and (17) and suppose that T satisfies conditions (20), (21) and (22). We then have a bubble equilibrium.*

It is clear from these trade dynamics that along the bubble equilibrium partially sophisticated investors generally get lower payoffs than fully rational investors. Yet, as emphasized above, we think of our equilibrium as the result of learning at historical level, whereby in the each bubble episode I-investors hold fresh cash and are not aware that their investment strategy is likely to result in a loss.

3.2.1 Example

While we postpone a more detailed analysis of conditions (20), (21) and (22) to Section 6.5 in the Appendix, we now highlight their structure with a numerical example. Suppose that $z = 0.2$, $r = 0.2$ and $K = 0.9$.

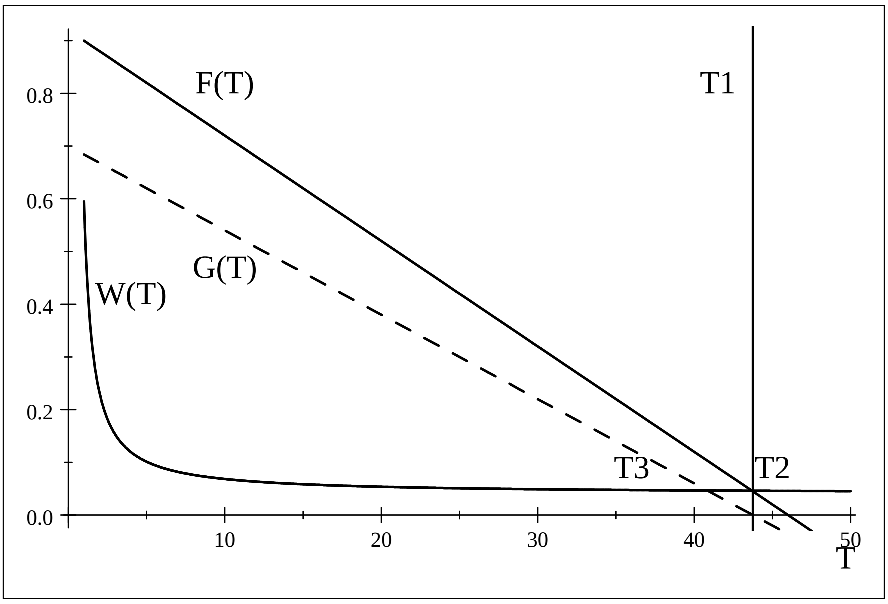

Figure 1: Conditions defining the duration of the bubble equilibrium for z=0.2, r=0.2 and K=0.9.

In Figure 1, the solid curve is the function $W(T)$, the solid line is the function $F(T) = K - z(1-K)(T-1)$, the dashed line is the function $G(T) = (1-r)[K - z(1-K)T] - r(1-K)$. $F(T)$ and $G(T)$ map the equilibrium T with the amount of investors who can buy in period T and $T+1$, respectively. These functions are derived in Section 6.5, and, by construction, they are such that $F(T) = K_T$ and $G(T) = K_{T+1}$. The vertical line plots $T = T_1$, as derived from condition (20). In this example, condition (20) is satisfied for $T \leq T_1$; condition (21) for $T \leq T_2$, where T_2 is defined by the intersection of $W(T)$ and $F(T)$; and condition (22) for $T > T_3$, where T_3 is defined by the intersection of $W(T)$ and $G(T)$. Specifically, substituting our values in equations (20), (21) and (22) we find that, up to integer approximations, they require respectively $T \leq 42$, $T \leq 43$, and $T \geq 41$. Hence, any $T \in \{41, 42\}$ can be a bubble equilibrium.

We now describe how investors' expectations evolve along the bubble equilibrium in this example. Consider the equilibrium in which all rational agents sell at $T = 42$. Fully rational agents have rational expectations, and so they expect the crash to occur at $t = 43$ throughout the duration of the bubble. On the contrary, as mentioned in the Introduction, I-agents' expectations may change once they observe the actual buy and sell rates realized in each period. On the one hand, after any $t < T$, I-agents revise their expectation about the date of the crash upwards. This is because, in all these periods, $\beta_t > \bar{\beta}$ and $\sigma_t < \bar{\sigma}$. On the other hand, at date T, I-agents revise their expectation about the date of the crash downwards (provided that r is not too small, see Section 4.2 for details). This is because $\beta_T < \bar{\beta}$ and $\sigma_T > \bar{\sigma}$.

These patterns are shown in Figure 2. The increasing solid line describes the actual supply observed along the equilibrium S_t (as determined by (17)). According to (19), this induces I-agents' expected supply $S_t^{I,s} = \bar{\sigma}(1-K)$ for each t and s. This expectation is represented by the constant dashed line S^I. The decreasing solid line describes the actual demand B_t (as determined by (16)). According to (7), (8) and (18), this induces I-agents' expected demand $B_t^{I,s} = \bar{\beta} K_t^{I,s}$ for each $t \geq s$. The decreasing dashed lines $B^{I,s}$ represent the period s expectations for I-agents about the demand in all periods $t \geq s$.

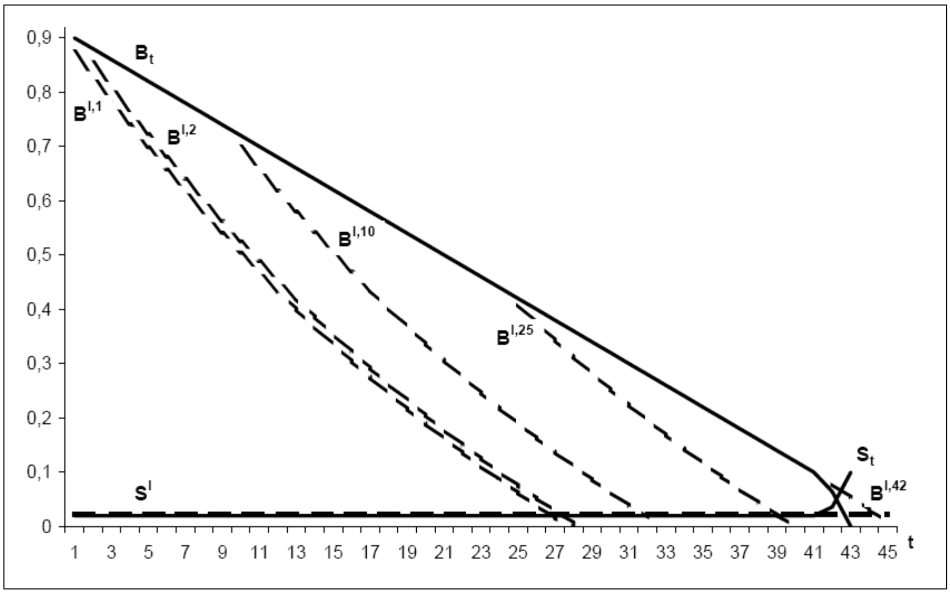

Figure 2: Equilibrium strategies and I-agents' expectations for z=0.2, r=0.2 and K=0.9.

The expected date of the crash in each period $s \leq T$, which is defined by $\min_t \left\{ t \mid B_t^{I,s} < S_t^{I,s} \right\}$, evolves accordingly. In this example, before any trade takes place, I-agents expect the crash to occur at $t = 27$. At the beginning of the second period, upon having observed fewer exits than expected, they expect the crash at $t = 28$. Similar dynamics occur at each $s \leq 42$: for example, at $s = 10$ they expect the crash at $t = 32$; at $s = 25$ they expect the crash at $t = 39$; at $s = 42$, when they buy for the last time, they expect the crash at $t = 45$. Only at the end of period 42, upon observing the massive exit by rational agents (and when it is too late to sell), they realize that the crash will indeed occur at $t = 43$.

3.3 Existence and maximal duration of a bubble equilibrium

We now investigate when a $T \geq 1$ satisfying conditions (20), (21) and (22) exists as a function of the parameters K, z and r. When such a T exists, we say that a bubble equilibrium exists. Intuitively, a bubble is more likely to develop when there is a large amount of cash that could potentially be used to fuel it; when not too many people are hit by shocks that force them to exit the speculative market, and when the number of investors who can correctly predict the date of the crash is not too large. We express this in the following Proposition.

Proposition 3 *There exists a $K^*(r,z) < 1$ such that if $K \geq K^*(r,z)$, then a bubble equilibrium exists. Such minimal $K^*(r,z)$ increases in r and z.*

As shown in the previous example, and more generally in Section 6.5, there need not be only one T satisfying conditions (20), (21) and (22). One natural point of interest is the largest T that can be sustained in equilibrium, the one which maximizes R-agents' profits.

This largest T is defined by conditions (20) and (21). The first condition can be explained by recalling that, even if no one exits voluntarily from the market, a mass $z(1-K)$ of agents sells in each period due to liquidity shocks. Hence, given that the mass of potential buyers decreases over time, R-agents must not exit too late if they want to find enough I-agents who buy their stocks. Condition (20) can be written as

$$T \leq \frac{K-r}{z(1-K)(1-r)} \equiv T_1. \tag{23}$$

Condition (21) instead imposes an upper bound on T whereby, if R-agents sell too late, I-agents would not buy, since the amount of cash observed at that stage would be too low. Such an upper bound is defined by

$$T \leq T_2,$$

where T_2 is the largest root solving $K - z(1-K)(T-1) = W$ (see Section 6.5 for details). Accordingly, we define the longest bubble equilibrium as

$$T_{\max} \equiv \min\{T_1, T_2\}.$$

In order to investigate how $T_{\max}$ varies with our exogenous parameters, the first issue is under which conditions T_1 or T_2 is the constraint defining $T_{\max}$. As shown in Section 6.7, when the fraction of rational agents r is small, the latter constraint binds, while the opposite occurs when z or K are small.[24] Irrespective of this, however, the comparative statics are clear: both T_1 and T_2 increase in K and decrease in r and z, as we show in the next Proposition.

Proposition 4 *The maximal equilibrium bubble T_{max} increases in K and decreases with z and r. Moreover, $T_{\max} \to \infty$ as $z \to 0$.*

Propositions 3 and 4 show that bubbles are supported by large K, small z and small r. These relations are consistent with empirical evidence. The effect of a large K is in line with the observation that speculative stocks tend to be in short supply initially, and that bubbles are sustained by the large involvement of new investors (see Cochrane, 2002; Kindleberger, 2005). A small probability of shock z implies that the fraction of potential investors decreases slowly, which is consistent with the fact that bubbles tend to display slow booms and sudden crashes (see Veldkamp, 2005). Finally, the effect of a small r echoes the observation that bubble episodes tend to attract a large number of inexperienced investors (see Shleifer, 2000; Kindleberger, 2005). However, as we discuss in the next section, the relation between bubbles and rationality is not so clear-cut, once we allow for uncertainty aversion.

4 Bubbles and rationality

In this section we discuss further the relation between bubbles and rational investors. We first show a sense in which rational investors need not be too many nor too few in our bubble equilibrium. We then explore how the maximal duration of a bubble varies with the share of rational agents in the market. Our model belongs to a class of models in which rational agents may not have the incentive to immediately stabilize the market. These models include Abreu and Brunnermeier (2003) and De Long et al. (1990b), who however differ in their predictions. In Abreu and Brunnermeier (2003), increasing the share of rational agents reduces the maximal bubble as it

[24] This observation implies that the equilibrium duration of the bubble is always strictly smaller than the largest possible bubble, $K/(z(1-K))$, as determined by the exogenous liquidity sales. In fact, when $r = 0$, $T_1 = K/(z(1-K))$ but $T_{\max} = T_2$.

reduces the buying capacity of irrational agents. Conversely, in De Long et al. (1990b), increasing the share of rational agents increases the size of the bubble as it distorts irrational agents' expectations. In our model, the relation between the maximal bubble and the share of rational investors can go both ways, and it depends crucially on the attitude of investors toward uncertainty. While Proposition 4 showed that increasing the share of rational investors reduces the maximal bubble when investors disregard uncertainty, Section 4.3 shows that the relation may be reversed if investors are sufficiently uncertainty-averse.

4.1 Rational investors should not be too numerous

As in standard models, we cannot have bubbles if all investors are fully rational. In particular, in a bubble equilibrium, r has to be small enough so that all rational agents are able to sell at T. This condition defines T_1, as expressed in equation (23). As we must have $T_1 \geq 1$, we need

$$r \leq \frac{K - z(1-K)}{1 - z(1-K)} \equiv r_{\max}.$$

Hence, we can define a necessary condition for the existence of a bubble equilibrium.

Proposition 5 *There exists a* $r_{\max} < 1$ *such that if* $r > r_{\max}$, *then no bubble equilibrium exists.*

4.2 Rational investors should not be too few

As expressed in Proposition 3, bubbles are more likely to arise when the share of rational investors r is low. On the other hand, in the bubble equilibrium characterized above, rational investors play a key role. By exiting at T, they give a negative shock to the market, which makes I-investors aware that they had overestimated the duration of the bubble and that the crash is about to occur. As a result, I-investors rush to sell as they realize everyone else is trying to sell. This final panic phase is a rather common feature of market crashes (see Kindleberger, 2005), and we now show that it requires r to be not too small. Specifically, consider the following condition:

$$B_{T+1}^{I,T+1} < S_{T+1}^{I,T+1}, \tag{24}$$

which ensures that, at the beginning of $T+1$, just before the crash occurs, all investors expect the crash to occur in this period. Together with condition (21), this requires that I-investors' expectation about the date of the crash changes between period T and period $T+1$, which in turns requires that some bad shock occurs in period T. Since the only source of such bad shocks is that R-investors decide to exit, we need sufficiently many of them. We can state this more precisely with the following Proposition.

Proposition 6 *In a bubble equilibrium where I-agents, at the beginning of $T+1$, realize that the market will indeed burst at $T+1$, we must have*

$$r > r_{\min},$$

where $r_{\min}$ is implicitly defined by the condition $Tr_{\min} = 1$.

4.3 Uncertainty Aversion

Proposition 4 shows that bubbles are more likely to last longer when the fraction of rational investors is smaller. We now show that this need not be the case if we consider a setting with uncertainty-averse agents.[25]

In our model, uncertainty concerns solely the predictions of what other investors will do. Hence, the amount of uncertainty faced by each agent depends on his ability to understand other investors' equilibrium strategies. If some I-agent perceives enough uncertainty, and he prefers to avoid it, he may refrain from investing in the speculative market. On the other hand, since fully rational agents face no uncertainty, they may still be willing to invest. As a result, the amount of investors in the speculative market is in

[25] Uncertainty (or equivalently ambiguity) describes situations in which agents' perceptions need not be accurate enough to provide them with a unique probability measure over the possible states of the world.

general increasing with the share of rational agents. This in turn may induce more optimistic expectations and higher demand, thereby sustaining longer bubbles.

In order to formally illustrate this idea, we enrich our setting by assuming that, independently from their cognitive types, investors differ in their attitudes towards ambiguity. Such attitudes are not relevant for R-investors, however, since they face no ambiguity, as observed above. For I-investors, instead, we distinguish between ambiguity-averse investors H and ambiguity neutral investors L, which have mass $(1-r)h$ and $(1-r)(1-h)$ respectively. Admitting that their predictions can be mistaken by some ε, H-agents believe that, in every t, the actual buy rate β_t will be in the interval $[\bar{\beta}-\varepsilon, \bar{\beta}+\varepsilon] \cap [0,1]$ and the actual sell rate σ_t will be in the interval $[\bar{\sigma}-\varepsilon, \bar{\sigma}+\varepsilon] \cap [0,1]$.[26] Furthermore, these agents choose the optimal investment strategy by considering the worst realizations of β_t and σ_t.[27] Hence, in order to participate in the speculation, they require a return that compensates for the perceived uncertainty.[28] Investors of type L are instead neutral towards uncertainty (or, alternatively, they do not admit that their predictions can be mistaken). Hence, as in Section 2, such investors only consider the averages $\bar{\beta}$ and $\bar{\sigma}$.

Here we consider the special case of $z \to 0$ (Bianchi and Jehiel, 2008 provide a more general treatment). In this case, $T_{\max}$ is defined by T_1, which may increase in r since a higher r reduces the mass of ambiguity-averse agents in the market. In fact, the bubble equilibrium is such that H-agents exit at some $\tilde{T}$, by selling to L- and to R-agents; R-agents sell to L-agents in period $T > \tilde{T}$; and in period $T+1$ the crash occurs. The smaller the mass of H-agents, the smaller the amount of investors who buy stocks at $\tilde{T}$, and

[26] The error term ε is here taken as given. One could for example endogenize this interval by letting the expected β_t and σ_t lie between the minimum and the maximum buy and sell rates observed along the equilibrium.

[27] Formally, we are assuming that these investors have a set of probability measures over the possible realizations of β_t and σ_t. Investors compute the minimal expected payoffs conditional on each possible prior, and decide the investment strategy corresponding to the maximum of such payoffs. This idea, which may be thought as an extreme form of uncertainty aversion, was formalized by Gilboa and Schmeidler (1989).

[28] Indeed, many authors have invoked ambiguity aversion as a possible resolution of the Equity Premium Puzzle (Chen and Epstein, 2002; Klibanoff, Marinacci and Mukerji, 2005).

so the fewer are R-investors with stocks at T and the greater the amount of L-investors with cash at T. Hence, the lower is S_T, the higher is B_T, which pushes towards a higher $T_{\max}$. This result is expressed in the following Proposition.

Proposition 7 *If $z \to 0$, there exists a $\hat{r}(K,h) < 1$ such that $T_{\max}$ increases in r for every $r \leq \hat{r}$.*

5 Discussion and extensions

The above analysis has abstracted from several aspects of real world speculative phenomena. The aim was to highlight in the simplest way how expectations are formed and how they evolve, which is the main focus of our paper and we believe a central ingredient in speculative behaviors. We now discuss some ways in which our model can be enriched. We also discuss the robustness of our insights in these richer settings.

5.1 Extra randomness

Our economy evolves according to a deterministic pattern. In real world, extra randomness is likely to be at work. It is not difficult to accommodate our model to incorporate extra randomness. In addition to be worth investigating in its own right, such randomness may make inferences or information other than that based on the aggregate buy and sell rates less appealing to I-investors.

To illustrate, suppose that both demand and supply are affected by some stochastic component. For example, new investors with cash may appear and liquidity shocks may have different realizations in each period. The evolution of K_t in (4) can be written as $K_{n,t+1} = K_{n,t} - E_{n,t} + \varepsilon^K_{n,t}$ and the probability of liquidity shocks can be written as $z_{n,t} = z + \varepsilon^z_{n,t}$, where $\varepsilon^K_{n,t}$ and $\varepsilon^z_{n,t}$ are identically and independently distributed across bubble episodes n and periods of trade t and $E(\varepsilon^K_{n,t}) = E(\varepsilon^z_{n,t}) = 0$. As above, investors differ in their knowledge of past strategies. Fully rational investors know buy and sell rates which applied in each period of trade, as a function of the particular realizations of $\varepsilon^K_{n,t}$ and $\varepsilon^z_{n,t}$, while I-investors know $\bar{\beta}$ and $\bar{\sigma}$, averaged both across bubble episodes and across trade periods within a bubble episode. The mechanics of the bubble equilibrium are very similar to the previous analysis: I-investors initially observe an higher price increase than expected. They may attribute such a high price to a lower realization of $\varepsilon^z_{n,t}$ and to an higher realization of $\varepsilon^K_{n,t}$ with respect to their means, somehow believing that the current market situation is more favorable to bubbles than the average past market, while at the same time keeping their $\bar{\beta}$ and $\bar{\sigma}$ fixed. Their expectation about the date of the crash evolves in the same way as described above.

This setting displays however some important differences with respect to our previous analysis.[29] First, I-investors can make each observation of price and trade volume consistent with their expectations of $\bar{\beta}$ and $\bar{\sigma}$ by attributing them to a particular realization of $\varepsilon^K_{n,t}$ and $\varepsilon^z_{n,t}$. Second, in such a stochastic world, learning about strategies may be more meaningful than learning about the past realizations of T or of the maximum trading price.[30] In fact, while the latter crucially depend on the specificities of a given bubble episode, the fundamental issue that investors face in all these episodes lies in understanding how other investors' strategies evolve. Moreover, such an overall understanding of traders' strategies allows investors to update their expectations about future prices upon the observation of current market trends (as determined say by the realizations of $\varepsilon^K_{n,t}$ and $\varepsilon^z_{n,t}$). These observations can be viewed as providing a motivation for the focus on learning of investment strategies in our setting. A more systematic investigation of how different types of learning affect investment decisions and the functioning of the market remains in our view a very interesting avenue for future research.

[29] Notice also that introducing some equilibrium uncertainty may generate an endogenous limit in investors' optimal trading position, similarly to what we have exogenously imposed above by assuming that agents can hold at most one stock at a time.

[30] In a completely deterministic setting, instead, if I-agents know say the duration of past bubbles (which would be the same as in the current bubble), bubbles would not arise.

5.2 Increasing sophistication

The key mistake partially sophisticated investors make is in predicting when other investors stop buying. As already argued, such a prediction is likely not to be accurate as abrupt changes in investors' strategies tend to occur even with no major news, in days quite similar to days in which most investors wanted to buy. Nonetheless, I-investors' expectation of constant buy and sell rates throughout the bubble episode may be too extreme as well. Our results however can be derived with less extreme assumptions, allowing investors to distinguish somewhat between the various phases of the bubble.

Suppose for illustration that partially sophisticated investors distinguish two phases of the bubble: they expect some $\bar{\beta}$ and $\bar{\sigma}$ to occur in each $t < t^*$ and some other $\bar{\beta}'$ and $\bar{\sigma}'$ to occur at each $t \geq t^*$, where as above these expectations correspond to the true average strategies played within each stage of the bubble. Consider the bubble equilibrium described in equations (16) and (17), and suppose $t^* \leq T$. During the early stage of the bubble, I-agents' expectation is correct since strategies are indeed constant. The expectation in the second stage of the bubble depends on t^*. The higher is t^* the more precisely late strategies are perceived. In particular, the higher t^*, the less optimistic is I-agents' expectation and so the lower is the maximal sustainable bubble. However, to the extent that the strategies played in the last period of the bubble are perceived with some error, i.e. $t^* \leq T$, there exist some parameter values such that a bubble equilibrium exists.

5.3 Increasing heterogeneity

Bubbles are often characterized by a substantial heterogeneity of opinions, for example regarding when the crash occurs. This typically leads not only to high price volatility but also to high volumes of trade (Cochrane, 2002; Hong and Stein, 2007). Moreover, part of this trade volume may be induced by the fact that investors exit and then re-enter the market. In our framework, an account of this requires introducing further heterogeneity among investors. In line with our main research theme, investors would differ in their degree of sophistication as described by how many and which phases of the bubble they distinguish.

In such a setup, there would be several periods in which investors trade based on differences in opinions (as it happens at T in the previous analysis). Also, the order of exit from the speculative market would not necessarily be monotonic in the degree of sophistication. Some agents might decide to re-enter the speculative market after having exited, as the threshold property that characterizes optimal strategies in our main analysis (see Lemma 1) would hold within each phase that agents distinguish, but not necessarily for the entire duration of the market. We leave for future research to investigate whether such a setup can match the observed volatility and volume of trade as well as the observed price dynamics.

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6 Omitted proofs

6.1 Proof of Lemma 1

Note first that if agent $i \in \theta$ exits from the market at t, then he must expect the price to drop at $t+1$, i.e. $p^{\theta,t}_{t+1} < p^{\theta,t}_{t}$, otherwise he would rather exit at $t+1$. From equation (5), and the discussion thereafter, if $p^{\theta,t}_{t+1} < p^{\theta,t}_{t}$ then $K^{\theta,t}_{t+2} = 0$, so this agent expects the market to close at $t+2$. R-agents' expectation is correct, so the price indeed drops at $t+1$ and the market closes at $t+2$. Hence, these agents will not re-enter at $t+1$. Now consider I-agents. By equation (2), $p^{I,t}_{t+1} < p^{I,t}_{t}$ is equivalent to $B^{I,t}_{t+1} < S^{I,t}_{t+1}$, which, given equation (11), writes as $\bar{\beta}K^{I,t}_{t+1} < \bar{\sigma}(1-K)$. By equation (5), we have $K^{I,t}_{t+1} = \bar{\beta}K_t - \min\{\bar{\beta}K_t, \bar{\sigma}(1-K)\}$, so $p^{I,t}_{t+1} < p^{I,t}_{t}$ is equivalent to

$$K_t < \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2}.$$

Now, since K_t cannot increase over time, it must be that

$$K_s < \frac{\bar{\sigma}(1-K)(1+\bar{\beta})}{\bar{\beta}^2} \text{ for every } s \geq t.$$

This implies that, at any $s \geq t$, I-agents expect the price to drop at $s+1$ and the market to close at $s+2$. Hence, such agents will never enter again.

6.2 Proof of Proposition 1

If agent $i \in \theta$ expects $p^{\theta,t}_{t+1} \geq p^{\theta,t}_{t}$ he will buy at t since the strategy of buying at t and selling at $t+1$ gives a positive expected profit. Note that, for this reason, the proposed strategy is optimal even though the agent may be hit by a liquidity shock which forces him to sell at $t+1$. Conversely, if $p^{\theta,t}_{t+1} < p^{\theta,t}_{t}$, then $B^{\theta,t}_{t+1} < S^{\theta,t}_{t+1}$. By equation (5), this implies $K^{\theta,t}_{t+2} = 0$, so the agent expects the market to close at $t+2$. Hence, given that the agent expects that selling at $t+1$ would be unprofitable and selling after $t+2$ would be impossible, he does not buy at t. Hence, $p^{\theta,t}_{t+1} \geq p^{\theta,t}_{t}$ is also necessary for $i \in \theta$ to buy/stay in the market at t. Finally, as already noted in (2), equations (12) and (13) are equivalent.

6.3 Proof of Corollary 1

According to equation (11), condition (13) can be written as

$$\bar{\beta} K_{t+1}^{I,t} \geq \bar{\sigma}(1-K), \tag{25}$$

where by (5) we have $K_{t+1}^{I,t} = \bar{\beta} K_t - \min\{\bar{\beta} K_t, \bar{\sigma}(1-K)\}$. If $\bar{\beta} K_t < \bar{\sigma}(1-K)$, the agent would expect the price to drop at t and he would exit. This corresponds to condition (14), since $\bar{\sigma}(1-K)/\bar{\beta} < W$ and so $\bar{\beta} K_t < \bar{\sigma}(1-K)$ implies $K_t < W$. If instead $\bar{\beta} K_t \geq \bar{\sigma}(1-K)$, then $K_{t+1}^{I,t} = \bar{\beta} K_t - \bar{\sigma}(1-K)$. Substituting into (25) gives the result.

6.4 Proof of Lemma 2

Given that the exits from the speculative market are permanent (Lemma 1), if i wants to sell/stay out at t, then he wants to sell/stay out for every $s > t$. Likewise, if i wants to buy/stay in the market at t, this reveals that he wanted to buy/stay in the market at each $s < t$. Hence, an agent's strategy is simply described by the time at which he plans to exit. As we focus on pure strategy equilibria, all agents of a given type decide to exit at the same time. Let s and s' denote the exit period for R- and I-agents, respectively. We now show that we must have $s' = s + 1$. Firstly, note that $s < s$, as exiting with or after I-agents would imply not being able to sell and so incurring a loss. Secondly, assume that $s' = s + w$, with $w \geq 2$, which means that I-agents buy at $s + 1$. Hence, $B_{s+1} = K_{s+1}$ and $S_{s+1} = z(1 - K)$. By Corollary 1, I-agents buy at $s + 1$ only if $K_{s+1} \geq W$. However, since $\bar{\sigma} > z$ and $\bar{\beta} < 1$, $W > z(1 - K)$. Hence $K_{s+1} \geq W$ implies $B_{s+1} \geq S_{s+1}$, i.e. $p_{s+1} > p_s$. This contradicts the fact that R-agents prefer to exit at s, thus we must have $s' = s + 1$. Setting $s = T$ gives the result.

6.5 The conditions defining T

In order to express conditions (20), (21) and (22) in terms of our exogenous parameters, note first that, iterating equation (5), the amount of potential buyers in period s can be written as the difference between the initial amount of potential buyers K and the accumulated amount of exits up to period $s-1$, that is

$$K_s = K - \sum_{t=1}^{t=s-1} [V_t + (1-\beta_t)K_t]. \tag{26}$$

The volumes of trade induced by the bubble equilibrium are

$$V_t = \begin{cases} z(1-K) & \text{for } t \leq T-1, \\ z(1-K)(1-r) + (1-K)r & \text{for } t = T, \\ 0 & \text{for } t = T+1. \end{cases} \tag{27}$$

Hence, using equations (16), (26), (27) and rearranging terms, we get

$$K_T = K - z(1-K)(T-1),$$

and

$$K_{T+1} = (1-r)[K - z(1-K)T] - r(1-K).$$

Condition (20) requires $\beta_T K_T \geq \sigma_T(1-K)$. With simple algebra, it can be written as

$$T \leq \frac{K-r}{z(1-K)(1-r)} \equiv T_1.$$

We then turn to conditions (21) and (22). To see their structure, we first define the functions

$$F(T) \equiv K - z(1-K)(T-1),$$

and

$$G(T) \equiv (1-r)[K - z(1-K)T] - r(1-K),$$

where by construction $F(T) = K_T$ and $G(T) = K_{T+1}$. Note that these functions are decreasing in T and they both tend to minus infinity as T goes to infinity. Furthermore, with simple algebra, one can show that the function $W(T)$, as defined in equation (15) and in which $\bar{\beta}$ and $\bar{\sigma}$ are given by (18) and (19), is decreasing and convex in T, and that it tends to $2z(1-K)$ as T goes to infinity. Hence, both $F(T)$ and $G(T)$ can intersect $W(T)$ at most twice in $\mathbb{R}_+$.

Suppose indeed that both $F(T)$ and $G(T)$ intersect $W(T)$ twice. Let T_5 and T_2 be the roots solving $F(T_5) = W(T_5)$ and $F(T_2) = W(T_2)$, with $T_5 < T_2$; and similarly let T_4 and T_3 be the roots solving $G(T_4) = W(T_4)$ and $G(T_3) = W(T_3)$, with $T_4 < T_3$. Since $G(T) < F(T)$ for every T, we then have that $T_2 > T_3 > T_4 > T_5$. In this case, the bubble equilibrium writes as $T \in [T_5, T_4) \cup (T_3, T_{\max}]$, where $T_{\max} \equiv \min\{T_1, T_2\}$.

The possibility of two disjoint intervals defining the bubble equilibrium depends on the fact that, in our model, both the number of potential buyers at T and I-agents' expectation depends on T, as expressed by the functions $F(T)$, $G(T)$ and $W(T)$. If $F(T)$ and $G(T)$ were constant (i.e. if z were zero), then we would only have equilibria of the type $[T_5, T_4)$. According to condition (21), we would need $T \geq T_5$ in order to make I-agents' expectation sufficiently optimistic and induce them to buy (recall that $W(T)$ is decreasing, i.e. I-agents' optimism increases in T). On the other hand, condition (22) would require $T < T_4$: R-agents could not sell too late otherwise I-agents' expectation would be too optimistic and they would never sell, so the crash would not occur.

Conversely, if $W(T)$ were constant, we would only have equilibria of the type $(T_3, T_{\max}]$. Condition (21) would require that $T \leq T_2$. If R-agents sell too late, I-agents would not buy since the amount of cash observed at that stage would be too low. On the other hand, condition (22) requires $T > T_3$. If R-agents sell too early, I-agents would not exit at $T + 1$, so the crash would not occur. Hence, it would be optimal to stay in the market rather than selling at T.

As one would expect, equilibria of the type $[T_5, T_4)$ occur when $F(T)$ and $G(T)$ are very high, so the binding constraint is the evolution of I-agents' expectation; while equilibria of the type $(T_3, T_{\max}]$ occur when $F(T)$ and $G(T)$ are very low, so the binding constraint is the evolution of the amount of cash in the economy. Indeed, for K sufficiently high, equilibria of the type $[T_5, T_4)$ do not exist, since we have $T_4 < 1$ (as in Example 3.2.1). More generally, depending on the value of K, r and z, such T_2, T_3, T_4, T_5 may not exist or their value may be less than one. This means that the constraints defined above may or may not bind.

Rather than providing a full treatment of such T_2, T_3, T_4, T_5, our analysis was mainly interested in defining conditions for the existence of equilibrium bubble (as expressed in Proposition 3 and in Section 4.1) and in characterizing the comparative statics on the maximal equilibrium bubble $T_{\max}$ (as expressed in Proposition 4 and in Section 4.3).

6.6 Proof of Proposition 3

Note first that, for every K, z and r, we have $T_3 < T_{\max} \equiv \min\{T_1, T_2\}$. In fact, since $G(T) < F(T)$ for every T, we have that $T_3 < T_2$. Moreover, by definition, $G(T_1) = 0$, so condition (22) holds for sure at T_1 and then $T_3 < T_1$. Given the shape of the function $W(T)$ described in Section 6.5, the bubble equilibrium exists if and only if $W(T)$ and $F(T)$ intersect at least once, i.e. if there exists a $T_2 \geq 1$ such that $F(T_2) = W(T_2)$. In fact, when this is the case, $T_{\max}$ can always be sustained as equilibrium. Hence, a sufficient condition for the existence of a bubble equilibrium is that $W(T)$ and $F(T)$ intersect once and only once, which is the case when $K \geq W(1)$. With some algebra, we can write

$$K \geq W(1) \Longleftrightarrow K \geq \frac{[z(1-K)(1-r)+(1-K)(1+r)](3-r)}{(1-r)^2}. \tag{28}$$

Condition (28) can be rearranged to define a K^* such that if $K \geq K^*$ then $K \geq W(1)$, and so a bubble equilibrium exists. Moreover, one can see that such K^* is always smaller than one, and it increases in r and z.

6.7 The conditions defining $T_{\max}$

We now turn to the analysis of the conditions under which T_1 or T_2 defines $T_{\max} \equiv \min\{T_1, T_2\}$. Note first that $T_1 < T_2$ if and only if $W(T_1) < F(T_1)$. By definition of T_1, $F(T_1)(1-r) = S_T$ and $S_T = z(1-K)(1-r) + r(1-K)$, so $W(T_1) < F(T_1)$ can be written

$$\frac{z(1-K)(1-r)+r(1-K)}{1-r} > W(T_1). \tag{29}$$

In Section 3.3, we claimed that $T_{max} = T_2$ when r is small, and $T_{max} = T_1$ when z or K are small. We now show that this is indeed the case. Consider the first claim. Rearranging condition (29), we can define a threshold $\bar{r}$ such that $T_1 < T_2$ if and only if $r > \bar{r}$. Such threshold is implicitly defined by $\bar{r} = P(\bar{r})$, where

$$P(r) \equiv \frac{W(T_1) - z(1-K)}{W(T_1) + (1-z)(1-K)}. \tag{30}$$

In fact, $P(r)$ is increasing in $W(T_1)$, and $W(T_1)$ is increasing in r. Moreover, $P(0) > 0$ and $P(1) < 1$. Hence $r > P(r)$ holds for $r > \bar{r}$, where $\bar{r}$ is uniquely defined by $\bar{r} = P(\bar{r})$.

Now consider the case of $z \to 0$, i.e. the probability of liquidity shocks is very small. Both T_1 and T_2 tend to infinity as z tends to zero, but T_2 exceeds T_1. In fact if $z \to 0$, then $z(1-K) \to 0$, $T_1 \to \infty$ and $W(T_1) \to 0$. Hence, $P(r) \to 0$, so r always exceeds $P(r)$ and $T_{\max} = T_1$.

Finally, consider the conditions on K. Condition (29) can be rearranged as

$$K < \frac{r + (1-r)[z - W(T_1)]}{z(1-r) + r} \equiv Q(K).$$

Notice first that if $K = 1$, then $W(T_1) = 0$ and so $Q(1) = 1$. That is, if $K = 1$, then $T_1 = T_2$. For $K = 0$, no bubble equilibrium exists, so we only have to consider $K \geq K_{\min}$, where $K_{\min}$ corresponds to the case $T_1 = 1$ and it writes as

$$K_{\min} \equiv \frac{r + z(1-r)}{1 + z(1-r)}.$$

Now, it can be shown (with simple algebra) that $Q(K_{\min}) > K_{\min}$, which means that $T_1 < T_2$ for $K = K_{\min}$.

6.8 Proof of Proposition 4

By differentiating equation $T_1 = (K-r)/[z(1-K)(1-r)]$, we can see that T_1 increases in K and decreases with z and r. To see the effects on T_2, define the function $L(T) \equiv F(T) - W(T)$. By definition, $L(T_2) \equiv 0$. Differentiating the function $L(T)$, we can see that it decreases in T_2, z and r and it increases in K. Hence, by the implicit function theorem, T_2 increases in K and decreases with z and r. The second part of the Proposition can be shown by noting that both T_1 and T_2 tend to infinity as $z \to 0$.

6.9 Proof of Proposition 6

Condition (24) can be written $\bar{\beta} K_{T+1} < \bar{\sigma}(1-K)$. Recall that condition (21) requires $\bar{\beta} K_{T+1}^{I,T} \geq \bar{\sigma}(1-K)$. Hence, conditions (24) and (21) jointly require $K_{T+1} < K_{T+1}^{I,T}$. Recall that $K_{T+1} = \beta_T K_T - S_T$, and $K_{T+1}^{I,T} = \bar{\beta} K_T - \bar{\sigma}(1-K)$. Hence, $K_{T+1} < K_{T+1}^{I,T}$ if and only if

$$(\beta_T - \bar{\beta}) K_T + [\bar{\sigma}(1-K) - S_T] < 0. \tag{31}$$

Consider the first term in (31). Recall that $\beta_T = (1-r)$ and $\bar{\beta} = (T-r)/(T+1)$, so $\beta_T < \bar{\beta}$ requires $(T+1)(1-r) < (T-r)$, that is $rT > 1$. Now consider the second term in equation (31). Recall that $\bar{\sigma} = ((T-r)z + 1 + r)/(T+1)$ and $S_T = r(1-K) + z(1-K)(1-r)$. Hence, $\bar{\sigma}(1-K) < S_T$ requires $r(1-K) - z(1-K)Tr > 1 - K - z(1-K)$, that is $rT > 1$. Hence, condition (31) is satisfied if and only if $rT > 1$. In particular, recall that we must have $T \leq T_1$, where $T_1 = (K-r)/[z(1-K)(1-r)]$, so condition (31) requires $r > [z(1-K)(1-r)]/(K-r)$. Doing the algebra, the last inequality is satisfied for $r \in (r_1, r_2)$, where $r_1 > 0$. Hence, there exists a $r_{\min} > r_1 > 0$ such that if $r \leq r_{\min}$ condition (31) cannot hold.

6.10 Proof of Proposition 7

Note first that, as in equation (21), L-investors buy/stay in at t if and only if $K_t \geq W$, while H-investors buy/stay in at t if and only if $K_t \geq W(\varepsilon)$, where, by replacing $\bar{\beta}$ with $\bar{\beta} - \varepsilon$, and $\bar{\sigma}$ with $\bar{\sigma} + \varepsilon$ in (21), we have

$$W(\varepsilon) \equiv \frac{(1-K)(\bar{\sigma}+\varepsilon)(1+\bar{\beta}-\varepsilon)}{\left(\bar{\beta}-\varepsilon\right)^2}.$$

Since $W(\varepsilon)$ increases in ε, H-investors always sell before L-investors. We then define an equilibrium in which for $t < \tilde{T}$ no investor wants to exit (apart from the liquidity traders); H-investors leave the market at $\tilde{T}$, selling to rational agents R and to low ambiguity-averse agents L. At $T > \tilde{T}$, rational investors sell to L-agents. At $T+1$, L-agents realize that the crash is about to occur and they want to sell, while no one is willing to buy. The crash occurs and the market closes.

We are interested in how $T_{\max}$ varies with r, for a given proportion of ambiguity-averse agents h. Consider the effects on T_1, recalling that T_1 is defined by $B_T = S_T$. In our equilibrium, S_T includes all R-investors with stocks at T and the exogenous sales $z(1-K)$, while B_T includes all L-investors with cash at T. That is, $S_T = \sigma_t(1-K)$, where

$$\sigma_t = z + (1-z)\frac{r}{1-h(1-r)},$$

and $B_T = \beta_t K_T$, where

$$\beta_t = \frac{(1-r)(1-h)}{1-h(1-r)},$$

and

$$K_T = K - z(1-K)(T-1) - (1-r)h[(1-z)(1-K) + K - z(1-K)(\tilde{T}-1)].$$

In order to simplify the exposition, from now on we assume that H-investors perceive enough uncertainty to be induced to sell immediately, i.e. that ε is large enough to have $\tilde{T} = 1$.[31] Hence, $B_T \geq S_T$ defines the condition $T \leq T_1$, where

$$\begin{aligned} T_1 \equiv \frac{1}{z(1-K)}[&K + z(1-K) - (1-z+zK)(1-r)h \\ &- \frac{z(1-K) - hz(1-K)(1-r) - (1-z)(1-K)r}{(1-r)(1-h)}]. \end{aligned} \tag{32}$$

After simple algebra, we can see that

$$\frac{\partial T_1}{\partial r} = \frac{1}{z(1-K)}[h(1-z+zK) - \frac{1-K}{(1-r)^2(1-h)}], \tag{33}$$

[31] For example H-investors may think that β_t and σ_t are respectively drawn by distributions with mean $\bar{\beta}$ and $\bar{\sigma}$ and support on $[0,1]$. As they are extremely ambiguity-adverse, they assume $\beta_t = 0$ and $\sigma_t = 1$ for all t, so they exit as soon as possible. In other words, given that there is a one-to-one mapping between $\tilde{T}$ and ε, we now consider $\tilde{T}$ as an exogenous parameter of the model.

which is positive when

$$r \leq 1 - \sqrt{\frac{1-K}{h(1-h)(1-z+zK)}}. \tag{34}$$

Proposition 7 claims that if $z \to 0$, then $T_{\max}$ increases in r for every $r \leq \hat{r}$, where

$$\hat{r} \equiv 1 - \sqrt{\frac{1-K}{h(1-h)}}.$$

To see that, note first that $T_{\max} = T_1$ when $z \to 0$. In fact, one can replicate the analysis of Section 6.7 in the setting with ambiguity aversion and write that $T_1 < T_2$ if and only if r exceeds a threshold implicitly defined by

$$r > \frac{(1-h)[W(T_1) - z(1-K)]}{(1-h)[W(T_1) - z(1-K)] + 1 - K}.$$

If $z \to 0$ the right hand side of the last equation tends to zero, and so $T_1 < T_2$. Substituting $z = 0$ into equation (34) gives the result.