The Coalitional Nash Bargaining Solution*

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Abstract

The coalitional Nash bargaining solution is defined to be the core allocation for which the product of players' payoffs is maximal. We consider a non-cooperative model with discounting in which one team may form and every player is randomly selected to make a proposal in every period. The grand team, consisting of all players, generates the largest surplus. But a smaller team may form. We show that as players get more patient if an efficient and stationary equilibrium exists, it must deliver payoffs that correspond to the coalitional Nash bargaining solution. We also characterize when an efficient and stationary equilibrium exists, which requires conditions that go beyond the non-emptiness of the core.

1 Introduction

We study a non-cooperative model of bargaining in which one and only one team may form. The team that forms generates a surplus or revenue that depends on the composition of the team. A team forms when it agrees on how to share the surplus or revenue it generates. The grand team consisting of all players is the one that generates the largest surplus. However, a team (or coalition) smaller than the grand team may form and players left outside the

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winning team get 0. Players bargain under the threat that a team smaller than the grand team may form.[1]

We model the team formation game as a bargaining game in which in each period, every player has an equal chance of being selected to make an offer (about a team and sharing device). If the offer is accepted by all tentative team members, it is implemented and bargaining stops. Otherwise, one proceeds to the next stage, which has the same structure. Every player discounts future payoffs according to the same discount factor assumed to be close to 1.

We ask: (1) Under what conditions does there exist a stationary equilibrium that is almost efficient in the sense that the grand team almost always forms instantaneously - we refer to such an equilibrium as an asymptotically efficient equilibrium? (2) When an (asymptotically) efficient equilibrium exists, what are the resulting payoffs obtained by the various players?

We obtain the following characterizations. In an (asymptotically) efficient stationary equilibrium, the profile of payoffs lies in the core,[2] and the product of players' payoffs (or Nash product) is maximal among all core allocations. Moreover, an (asymptotically) efficient stationary equilibrium exists if and only if (1) the core is non-empty and (2) as one increases uniformly (i.e. by the same (small) amount Δ) the value of every team S, the maximal Nash product obtained over core allocations increases as well.

The first condition is familiar.[3] The second condition however has no counter part in the literature. We illustrate through examples that this second condition may fail even when the core has a non-empty interior. In particular, when all allocations in the core force some players to receive very small payoffs and when there is a gap between the surplus generated by the grand team and the surplus generated by any other team then there must be (significant) inefficiencies in any stationary equilibrium with patient players.

We shall refer to the allocation that maximizes the Nash product among all core allocations as the *coalitional Nash bargaining solution*. It can be viewed as a multi-player extension

[1] Teams are thus exclusive of each other. To justify that only one team can form, one can have in mind situations in which if more than one team forms, then following the team formation stage, there is competition between teams (think of Bertrand competition between teams in an oligopolistic market). To the extent that competition is harsh enough (think of similar marginal costs, and heterogenous fixed costs), then in equilibrium only one team can form.

[2] That is, it is such that every team (or coalition) S receives at least the revenue $v(S)$ generated by S and the grand team (or grand coalition) N receives no more than the feasible revenue $v(N)$.

[3] In an efficient equilibrium, any player must be making offers to the grand team only. If the core is empty, then there must exist a team S that generates a surplus larger than the sum of equilibrium payoffs accrued to that team. Thus any member of that team S would be better off making an offer to S rather than the grand team.

of the (two-player) Nash bargaining solution. As in Binmore et al. (1986), the connection between the non-cooperative bargaining game and the Nash maximization program is established by comparing the equilibrium conditions of the bargaining game[4] to the first order conditions characterizing the solution of the maximization program.[5] Note that in the coalitional Nash bargaining solution, we consider the maximization of the product of players' payoffs, as opposed to the product of payoffs that would each be deflated by some reservation value. This is reminiscent of the finding in Binmore et al. (1986) in the two-person case that the relevant 'threat point' in Nash theory should be the status quo payoff, i.e. the payoff obtained by players when one period (with no agreement) elapses.

From another perspective, the finding that only a subset of coalitions matters is related to the idea that not all teams (or coalitions) can *credibly* form in equilibrium. Those coalitions that pin down the coalitional Nash bargaining solution are those teams that can credibly form in equilibrium. Our analysis can thus be viewed as generalizing the insight due to Binmore et al. (1989) about the role of outside options in two-person bargaining: Binmore at al. (1989) pointed out that outside options should have no effect on the equilibrium outcome if they are not credible (because they deliver payoffs that are inferior to the equilibrium payoffs obtained when outside options are absent). In our analysis too, not all teams can credibly form and our characterization allows us to understand simply which teams can credibly form in equilibrium.

Related literature on coalition formation:

There is a vast literature on coalition formation. An excellent overview of this literature appears in Ray (2008)'s book. As explained in Ray (2008), the recent literature on coalition formation has mostly considered non-cooperative bargaining approaches, as in our paper.[6] One focus of the recent literature has been on the possibility of externalities between coalitions (see, in particular, Bloch (1996) and Ray and Vohra (1999)). Our paper does not consider general forms of coalitional externalities even if our assumption that coalitions are exclusive of each other (i.e., only one team can form) can be interpreted from this more general perspective.[7]

[4] i.e. the equilibrium conditions derived from looking at one shot deviations.

[5] Other bargaining models obtain a link between non-cooperative outcomes and the multi-person Nash bargaining solution (see in particular Krishna and Serrano (1996) in which each agent can sell his rights to another agent and exit the game). These papers however do not consider the possibility of forming subcoalitions.

[6] The literature has also developed what Ray (2008) refers to as the blocking coalition approach in which moves by coalitions instead of individuals are considered.

[7] See footnote 1, and also Selten (1981) for the first paper of this non-cooperative multi-player bargaining

Our paper belongs to the literature that examines bargaining protocols in which agreements to form coalitions are irreversible (chapter 7 of Ray's book) and in which players (even though patient) discount future payoffs.[8] The first paper in this strand is Chatterjee et al. (1993).[9] They consider situations in which the first player to reject an offer (in a tentative agreement to form a coalition) is also the next player to make an offer,[10] and in which those players who are left outside a forming coalition can continue bargaining to form other coalitions (Chatterjee et al. assume that there are no coalitional externalities). An important finding of Chatterjee et al. is that whenever the egalitarian solution in which every player gets the same share of the grand coalition value is not in the core, efficiency cannot be guaranteed irrespective of who the first proposer is.[11] Chatterjee et al. proceed from there by analyzing when efficiency can be obtained for some well chosen first proposer, and they show that this is so for strictly convex games (i.e., games such that $v(S \cup T) > v(S) + v(T) - v(S \cap T)$).[12]

In line with Chatterjee et al., we examine the rejector-proposes protocol in our setting where only one team can form (see Section 6). We observe too that asymptotic inefficiency cannot obtain when the egalitarian solution is not in the core (assuming the first proposer is chosen at random). Thus, our analysis of the random offer proposer model -which establishes that asymptotic efficiency obtains beyond the situations in which the egalitarian solution lies in the core-[13] allows us to identify the extra source of inefficiency imposed by the rejector-proposes protocol – in which (too much) bargaining power (to propose) is given to the player that is approached and rejects the tentative agreement proposal.

Our analysis also completes the insight obtained by Okada (1996) who considered the

literature which also makes this assumption.

[8] For models with reversible agreements, see Konishi and Ray (2003) for a blocking coalition approach and Gomes and Jehiel (2005) for a non-cooperative bargaining approach.

[9] Other papers in this strand not considering discounting and as a result not obtaining inefficiencies include Selten (1981), Perry and Reny (1994) and Moldovanu and Winter(1995).

[10] This assumption appears first in Selten (1981).

[11] In our setup too, if we consider the rejector-proposes protocol and any player can be the first proposer, we get inefficiencies as soon as the egalitarian solution is not in the core (see Proposition 4 in Section 6).

[12] The efficient outcome obtained then by Chatterjee et al. corresponds to the point of the core that is *Lorenz-maximal*. That solution happens to coincide with the coalitional Nash bargaining solution in convex games. Observe that the Lorenz-maximal selection of the core would be picked by the maximization of any symmetric quasi-convex function of the vector of individual payoffs (see Dutta and Ray 1989 for a formal statement relating the egalitarian core with the maximization of quasi-convex functions), so in particular by the maximization of the Nash product. It should also be mentioned that when the game is not convex, the Lorenz-maximal selection may not be unique.

[13] And yet not for all situations in which the core is non-empty.

random proposer protocol in Chatterjee et al.'s model, but yet restricted attention to stationary equilibria that employ pure strategies (such equilibria may fail to exist in general).[14] His main result is that if an efficient outcome obtains in pure strategies, it must coincide with the egalitarian solution, and this outcome may be obtained in equilibrium if and only if the egalitarian solution lies in the core.[15] What our characterization results show is that, in the random proposer protocol, asymptotic efficiency obtains for a broader range of characteristic functions, provided equilibria in mixed strategies are considered.[16] Moreover, considering mixed strategies allow us to understand which coalitions constitute credible threats in equilibrium for various characteristic functions (whereas the restriction to pure strategies implies that no subcoalition can credibly form).

In the rest of the paper, we present the model (Section 2), our characterization results (Section 3), applications of the coalitional Nash bargaining solution (Section 4), the proofs of our main results (Section 5) and a discussion including the connection to the rejector-proposes protocol and to other familiar solutions found in the cooperative game literature (Section 6).

2 The Model

We consider n players labelled $i = 1, .., n$. Any subset S of these players may form a team. The surplus that such a team generates is non-negative and denoted $v(S)$. Once a team forms the game stops and no further team can form. We let 0 be the payoff that player i obtains as long as no team has formed yet, where 0 is also the payoff player i gets when he is not part of the winning team. We denote by $N = \{1, ..., n\}$ the set that comprises all players, and by $\mathcal{S}$ the set of subsets of N.

The bargaining game. In any period $t = 1, ...,$ a proposer is selected randomly. Draws are independent, and each player i has an equal chance $1/n$ of being selected. A proposer chooses a subset S of agents, possibly equal to $N = \{1, .., n\}$. He makes a proposal $x^S = (x_i^S)_{i \in S}$ to

[14] Chatterjee et al. also restrict attention to stationary equilibria in pure strategies. Yet, in their setup in which the first agent to reject the offer is the next proposer, this does not perclude the existence of an equilibrium for strictly convex games.

[15] Observe that if the egalitarian solution lies in the core, it coincides with the coalitional Nash bargaining solution. Hence, Okada's insight agrees with ours in this special case.

[16] In a recent paper, Okada (2007) extends the definition of the stationary equilibrium in his earlier model (Okada, 1996) to allow for mixed strategies (existence is then ensured). Yet, his analysis of efficiency is still restricted to the case of stationary equilibria that employ pure strategies. At the end of chapter 7, Ray (2008) mentions that it would be worthwile to extend Okada's analysis to cope with mixed strategies. This is achieved by this paper, assuming one team can form.

share the surplus $v(S)$ that satisfies the feasibility condition:

$$\sum_i x_i^S \leq v(S)$$

Players in S are asked whether they accept or reject the proposal. If they all accept, the team S forms and each team member $i \in S$ gets a payoff equal to x_i^S (non-team members get 0). If one or more players in S rejects the proposal, the game moves to the next period, which has the same structure. All players discount future payoffs with the same discount factor $\delta < 1$. So in case a coalition S forms at date t and accepts x^S, ex ante payoffs are $\delta^{t-1} x_i^S$ if i belongs to the coalition S, or 0 if player i does not belong to S.

Throughout the paper, we assume that the grand team is the only efficient coalition. That is,

$$v(N) > v(S) \text{ for all } S \neq N.$$

We call $\mathcal{P}$ the set of possible proposals, and $\mathcal{A}$ the set of possible acceptance rules. In principle, a strategy for player i specifies, at each date t, and for every history of the game up to date t, a proposal $(S, x^S) \in \mathcal{P}$ (possibly a mixed strategy proposal) in case i is the proposer, and an acceptance rule in $\mathcal{A}$ in case i is not a proposer. We will however restrict our attention to *stationary* equilibria of this game, in which each player adopts the same (possibly mixed strategy) proposal and acceptance rule at all dates. We will analyze when such equilibria can be almost efficient as δ tends to 1, and we will characterize the payoffs obtained by the various players in such equilibria, in the limit as δ tends to 1.

We first introduce a few preliminary definitions. We define an allocation as a vector of payments $x = (x_1, ..., x_n)$, and we let $x(S) = \sum_{i \in S} x_i$. Given our assumption that $v(N) = \max_S v(S)$, an allocation x is *efficient* if

$$x(N) = v(N).$$

We turn now to the notion of core stability. An allocation x is *feasible* if

$$x(N) \leq v(N).$$

It cannot be blocked by a coalition S if

$$x(S) \geq v(S)$$

Definition 1. The *core* consists of the allocations x that are feasible and that cannot be blocked by any coalition $S \in \mathcal{S}$.

We denote by $\mathcal{C}$ the set of core allocations when it is non-empty. And we further define:

Definition 2. Assume the core is non-empty. The *coalitional Nash bargaining solution* is the core allocation x^* that maximizes the Nash product.[17] That is:

$$\{x^*\} = \arg\max_{x \in \mathcal{C}} \prod_{i \in N} x_i.$$

Definition 3. The bargaining game has an *asymptotically efficient equilibrium* with limit value u if there exists a sequence of pairs of discount factor and associated equilibrium value vector $(\delta^{(k)}, u^{(k)})$ with the property that $\delta^{(k)}$ tends to 1, $u^{(k)}$ tends to u, and $u(N) = v(N)$.

3 Main Results

3.1 A first necessary condition.

When the core is empty, there is no way one can sustain a stationary equilibrium that is approximately efficient as δ tends to 1.

Proposition 0. *When the core is empty there cannot exist an asymptotically efficient equilibrium.*

This is a fairly simple observation. Assume by contradiction that there exists an asymptotically efficient equilibrium with limit value u. When a coalition other than N is formed, there is an efficiency loss at least equal to $\min_{S \neq N} (v(N) - v(S)) > 0$, so coalitions other than N should almost never form. This implies that when a player, say i, is selected to make a proposal, he must be getting a continuation equilibrium payoff arbitrarily close to $v(N) - \sum_{j \neq i} u_j = u_i$.

Now observe that if the core is empty and $u(N) = v(N)$, there must exist a coalition S^* such that $v(S^*) > u(S^*)$. Thus, any player i in S^* when selected to make an offer can get a payoff arbitrarily close to $v(S^*) - u(S^* \setminus \{i\}) = u_i + v(S^*) - u(S^*)$ by approaching coalition S^*. Since $u_i + v(S^*) - u(S^*)$ is bounded away from u_i, it follows that $i \in S^*$ would not propose coalition N, thereby yielding a contradiction.

In line with the previous literature on coalition formation, the above claim asserts that the existence of an asymptotically efficient equilibrium requires the non-emptiness of the core. Note that the argument also implies that an asymptotically efficient equilibrium must have a limit value in the core.

[17] It is uniquely defined because the core is a convex set.

From now on, we will assume that the core is non-empty. Formally, we define v_* as the smallest value of $v(N)$ for which the core is non-empty:

$$v_* \equiv \min_{x_1 \ldots x_n} \{x(N) \mid x(S) \geq v(S), \forall S \neq N\}.$$

We shall assume:[18]

A1 (non-empty core): $v(N) > v_*$.

3.2 Does A1 guaranty asymptotic efficiency?

When players are homogeneous (thereby implying that the value of a coalition is solely determined by its size), then, as we shall see, the non-emptiness of the core implies the existence of an asymptotically efficient equilibrium:[19] the limit payoff profile obtained there corresponds to the egalitarian solution in which every player gets $\frac{v(N)}{n}$.

When players are heterogeneous, we shall see that A1 alone does not guarantee asymptotic efficiency. Example 1 below illustrates a case in which A1 does not ensure the existence of an asymptotically efficient equilibrium.

Example 1. Consider two groups of players A and B ($A \cup B = N$). There are a players in A and b players in B with $a + b = n$. The values of coalitions solely depend on the number of players in A and B. We assume that apart from the grand coalition the only relevant coalitions have n_a players in A and n_b players in B with $\frac{b}{n_b} > \frac{a}{n_a} > 1$. This means that subcoalitions require only a strict subset of either type of players (A or B), but that a larger fraction of players in A are needed.

Formally, we let $v(m_a, m_b)$ denote the value of a coalition with m_a players in A and m_b players in B. We have $v(a, b) = v(N)$. We let $v(n_a, n_b) = 1$ and assume that for $(m_a, m_b) \neq (a, b)$,

$$v(m_a, m_b) = \begin{cases} 0 \text{ if } m_a < n_a \text{ or } m_b < m_b \\ 1 \text{ if } m_a \geq n_a \text{ and } m_b \geq n_b \end{cases}$$

It is readily verified that $v_* = \frac{a}{n_a}$. Let

$$v^* = \frac{a(1 - \frac{n_b}{b}) + b(1 - \frac{n_a}{a})}{n_a(1 - \frac{n_b}{b}) + n_b(1 - \frac{n_a}{a})}$$

[18]Note that this is a condition that is slightly stronger than the non emptiness of the core (which just requires that $v(N) \geq v_*$).

[19]When δ is sufficiently close to 1, there even exists an equilibrium in pure strategies that achieves an efficient outcome.

which is larger than v_* (because $\frac{n_a}{n_b} > \frac{a}{b}$). For $v(N) < v_*$, the core is empty.

Our main characterization results will establish that (1) no asymptotically efficient equilibrium exists for all $v(N) \in (v_*, v^*)$ and (2) an asymptotically efficient equilibrium exists for all $v(N) \geq v^*$.

Let us here provide an intuition as to why asymptotic inefficiencies must prevail in Example 1 when $v(N)$ lies slightly above v_*. Observe that in this case, it should be that players in B get approximately zero for an allocation to be in the core. We suggest below that such a payoff cannot arise in equilibrium. Thus, since any asymptotically efficient equilibrium has a limit value in the core (see the proof of Proposition 0), inefficiencies must arise.

A feature of (stationary) equilibrium is that each player i is indifferent between accepting the equilibrium proposal he is offered and waiting one more period, or equivalently, the individual cost incurred by player i when waiting one period should be set equal to the share $\frac{1}{n}$ of social cost of waiting one more period in equilibrium. Call u_i the lowest payoff player i would accept in equilibrium, and let S denote the social cost of waiting one period. Consider first the case in which the grand coalition would always form immediately in equilibrium. Not accepting the offer u_i induces a waiting cost $(1-\delta)u_i$ for player i, but it also induces a chance $1/n$ of being the proposer next period, in which case player i can extract the entire social cost of waiting one period:

$$(1-\delta)u_i = \delta\frac{S}{n} \tag{1}$$

In the proposed equilibrium, $S = (1-\delta)v(N)$, which explains that $v(N)$ is then equally shared.

One way to reduce a player's equilibrium payoff is to ensure that he is not always part of the coalition that forms in equilibrium: this makes rejecting the offer u_i more costly as player i would run the risk of not being in the coalition that eventually forms. Call π the probability that player i is not in the coalition that forms. The cost of rejecting the offer u_i is now $u_i - \delta(1-\pi)u_i$, hence condition (1) becomes:

$$(1-\delta+\delta\pi)u_i = \frac{S}{n}$$

therefore potentially reducing the payoff that i obtains in equilibrium if π is chosen large compared to $(1-\delta)$. Yet S, the social cost of waiting, increases (linearly) with π as well: in any event where i is not part of the coalition that forms, only 1 instead of $v(N)$ is shared, thereby resulting in a social cost of $v(N) - 1 \geq v_* - 1$. It follows that player i's equilibrium payoff cannot get arbitrarily small. We will come back to this observation more formally after we present our main results.

3.3 The general case.

The findings that we report now consider arbitrary forms of heterogeneity among players.[20] First, we provide a tight characterization of the conditions required for the existence of an asymptotically efficient equilibrium. Second, when an asymptotically efficient equilibrium exists, we show that it must coincide with the coalitional Nash bargaining solution introduced in Section 2.

To state the condition for the existence of an asymptotically efficient equilibrium, it is convenient to define the Δ-core as the set of allocations x that satisfy the constraints:

$$\begin{aligned} x(N) &\leq v(N) - \Delta \\ x(S) &\geq v(S) - \Delta \text{ for all } S \in \mathcal{S} \end{aligned}$$

We denote it $\mathcal{C}(\Delta)$. This is the core of an economy in which the values of all coalitions are reduced by Δ.[21] In what follows, Δ will be a non-negative scalar. When we shall move to the equilibrium analysis of our coalition formation game, we will interpret Δ as the inefficiency that arises in equilibrium.

Let $\mathcal{N}(\Delta)$ denote the maximum of the Nash product among allocations in the $\Delta-$core, that is:

$$\mathcal{N}(\Delta) = \max_{x \in \mathcal{C}(\Delta)} \prod_{i \in N} x_i.$$

Increasing Δ from 0 has two effects. It reduces total welfare (measured as the sum of the utilities). This would seem to decrease the Nash product. But it also relaxes the constraints imposed by the coalitions. The resulting effect on the Nash product is thus ambiguous in general.

Whether equilibria can be efficient when players get very patient precisely depends on whether increasing Δ reduces the maximum Nash product $\mathcal{N}(\Delta)$ or not.

Formally, we define the following property:

> *P1: There exists* Δ_0 *such that for all* $\Delta \in (0, \Delta_0)$, $\mathcal{N}(\Delta)$ *is strictly decreasing in* Δ.

Our main results are stated in the following two propositions.

[20] Also note that our approach allows the players to employ any stationary strategies, pure or mixed.

[21] We could alternatively assume that only the values of coalitions S such that $v(S) > \Delta$ are reduced by Δ (while the value of other coalitions would be set at 0). Since only small Δ are considered, this is equivalent.

Proposition 1. *Let A1 hold. If P1 holds, then there exists an asymptotically efficient equilibrium. If P1 does not hold then all stationary equilibria of the bargaining game remain bounded away from efficiency.*

Proposition 2. *Let A1 hold. Assume there exists an asymptotically efficient equilibrium with limit value u. Then (P1 must hold and) u must coincide with the coalitional Nash bargaining solution.*

An asymptotically efficient equilibrium (when it exists) is thus *uniquely* determined by the maximization program defining the coalitional Nash bargaining solution. Whether it exists requires that the core be non-empty and that the maximal Nash product subject to the core constraints increases as the values of all coalitions is increased by the same (small) amount (condition $P1$).

3.4 Varying the value $v(N)$.

From an economic viewpoint, it is instructive to consider whether an asymptotically efficient equilibrium exists when we vary the value of the grand team N while keeping the values of all other coalitions $S \neq N$ fixed. Recall that v_* is the lowest value of $v(N)$ for which the core is non empty. We shall denote by v^E the smallest value of $v(N)$ for which the egalitarian solution (each player gets $\frac{v(N)}{N}$) lies in the core:

$$v_E = n \max_S \frac{v(S)}{|S|}$$

- When $v(N) < v_*$, we know by Proposition 0 that there is no asymptotically efficient equilibrium.

- When $v(N) \geq v^E$, then clearly the egalitarian solution is also the coalitional Nash bargaining solution, and condition $P1$ holds (because $\left(\frac{v(N)-\Delta}{n}\right)^n$ is decreasing in Δ). Hence Proposition 1 shows that an asymptotically efficient equilibrium exists.[22]

- When players are heterogeneous, we typically have that $v_* < v^E$ and we are thus left with the case in which $v_* < v(N) < v^E$. In this range, the core has a

[22]For $v(N) \geq v_E$, there even exists, for δ sufficiently close to 1, an equilibrium in pure strategies that is exactly efficient: the proposer makes an offer with probability 1 to the grand team; he offers $\delta\frac{v(N)}{n}$ to each other player while keeping the rest for himself and this is accepted.

This is essentially the insight developed in Okada (1996) (for the case in which left aside agents can continue bargaining to form further coalitions).

non-empty interior, but whether there is an asymptotically efficient equilibrium (in this range) depends on whether condition $P1$ holds or not.[23]

Technically, condition $P1$ can be checked just by looking at the derivative of the Nash product with respect to Δ in the above maximization program. Thus, it is a relatively simple condition to check for each characteristic function $v(\cdot)$, as example 1 illustrates.

The case of Example 1.

In example 1, it is readily verified that $v^E = \frac{n}{n_a+n_b}$ and $v_* = \frac{a}{n_a}$. For $v(N) \in (v_*, v^E)$, the allocation that maximizes the Nash product over all allocations $x \in \mathcal{C}(\Delta)$ gives x_a to each player in A and x_b to each player in B, with (x_a, x_b) such that $ax_a + bx_b = v(N) - \Delta$ and $n_a x_a + n_b x_b = 1 - \Delta$. It can be checked that $\mathcal{N}(\Delta)$ increases in Δ if and only if $v(N) < v^*$.

In the rest of this Section, we elaborate on when condition $P1$ holds or fails.

3.5 Conditions under which P1 fails.

We will show that $P1$ typically fails when two conditions hold: (1) core constraints imply that at least one player gets very small payoffs, and (2) there is a gap between v_* and the largest value a subcoalition can obtain. We start with the case in which the core is small (i.e. $v(N)$ close to v_*). We then generalize the condition obtained to larger values of $v(N)$.

Formally, we fix $v(S)$, $S \neq N$, and vary the value of $v(N)$. We denote by C^0 the core associated with $v(N) = v_*$, i.e. the smallest value of $v(N)$ for which the core is non-empty. We also denote by I_0 the set of players i_0 such that for all $x \in C^0$, $x_{i_0} = 0$. We define the following conditions:

(C1) $I_0 \neq \emptyset$.

(C2) $v_* > \max_{S \neq N} v(S)$

Theses conditions say that at the point where the core becomes non-empty, a subset I_0 must get a payoff equal to 0, and all subcoalitions are strictly inefficient as compared with the grand team. We have the following claim:[24]

[23] In this range, it can be shown that allowing for mixed strategies is important, as there exists no equilibrium in pure strategy for δ close enough to 1.

[24] Condition P1 could also fail for $v(N)$ slightly above v_* even if C2 fails. For example, consider the three player situation in which only the singleton coalitions are binding with $v(1) = a > v(2) = b > v(3) = c > 0$. Here $v_* = a + b + c$ and Condition P1 is violated for $v(N)$ slightly above v_* as soon as $1 > \frac{c}{a} + \frac{c}{b}$, that is, if c is sufficiently small relative to both a and b.

Claim A: Fix $v(S)$, $S \neq N$. If (C1) and (C2) holds, then there exists $\varepsilon > 0$ such that condition $P1$ fails whenever $v(N) \in (v_*, v_* + \varepsilon)$.

Intuition for the proof. Consider the set $I_1 = N \backslash I_0$, set $v(N)$ close to v_*, and consider the Nash product maximization program. Condition C2 implies that any coalition S containing I_1 cannot be binding (otherwise $v(S)$ would be close to v_*). So the size of any binding coalitions is *strictly* smaller than $|I_1|$. This in turn implies that whenever the values of all coalitions are reduced by some amount Δ, one can reduce the payoff of players in I_1 by an amount Δ_1 slightly larger than $\frac{\Delta}{|I_1|}$ and still satisfy the core constraints. The residual $|I_1| \Delta_1 - \Delta$ can then be distributed to players in I_0, thereby leading to an increase of the Nash product.

Example 1 (continued). Conditions (C1) and (C2) are satisfied in Example 1. Condition (C1) holds because at the point where the core becomes non-empty, all players in B get a payoff equal to 0. Condition (C2) holds because core constraints bind due to teams that do not include all players in $I_1 = N \backslash I_0 = A$, but only a subset of them ($n_a < a$); as a result, $v_* = \frac{n_a}{a} > 1 = \max_{S \subsetneq N} v(S)$.

Another interpretation of (C1) and (C2).
To interpret (C1) and (C2) in light of the coalition structure induced by the characteristic function v, consider the minimization program $\mathcal{P}$ defined by:

$$\arg \min_{x_1 \ldots x_n} \{x(N) \mid x(S) \geq v(S), \forall S \neq N\} \qquad (\mathcal{P})$$

and consider the subcoalitions S for which $v(S) > 0$ and which are binding in program $\mathcal{P}$. Call these coalitions *strong coalitions*. Also call *weak* a player that does not belong to any strong coalition.

- Condition (C1) holds when there exists a weak player.[25]

- Condition (C2) holds if and only if for all arbitrarily nearby characteristic functions, there are at least *two* strong subcoalitions.[26]

[25] If a player does not belong to any strong coalition, his payoff must be zero at $v(N) = v_*$. Otherwise, his payoff could be reduced locally while satisfying all no subcoalition blocking constraints, hence $x(N)$ could be reduced further, thus contradicting the definition of v_*.

[26] If condition (C2) does not hold, then $v_* = v(S_*)$ for some S_*. Hence, for some nearby characteristic function for which all $v(S) > 0$ except $v(S_*)$ are reduced, S_* is the only strong coalition.

Conversely, assume that for all ε, there exists a characteristic function v^ε ε−close to v with only one strong coalition, say S_*^ε. Then $v_*^\varepsilon = v^\varepsilon(S_*^\varepsilon) = \max_{S \neq N} v^\varepsilon(S)$. Since, as ε gets small, v_*^ε is arbitrarily close to v_*, and $\max_{S \neq N} v^\varepsilon(S)$ is arbitrarily close to $\max_{S \neq N} v(S)$, v_* is also arbitrarily close to $\max_{S \neq N} v(S)$. Hence (C2) does not hold.

Generalization. Finally, claim A can be generalized to values of $v(N)$ that may lie away from v_*. Let

$$\eta = v(N) - \max_{S \subsetneq N} v(S)$$

and define

$$H = \{x, x_i \geq \eta/N \text{ for all } i\}.$$

We have:

Claim A': If $H \cap C = \emptyset$, then $P1$ cannot hold.

The proof consists in showing that when $P1$ holds, each player must be getting a payoff at least equal to η/N, thereby making it impossible to have $P1$ when $H \cap C = \emptyset$. We will come back later to this result when we discuss why it is not possible *in an equilibrium of the bargaining game* that a player gets a very low payoff whenever there is a gap between the value of the grand team and the value of any other team: as we shall see, in an efficient equilibrium, any player (who has chance $1/N$ of being selected as a proposer) can secure a payoff at least equal to η/N. This will provide another intuition for the inefficiencies that must arise under the conditions of claim A'.

3.6 Conditions under which P1 holds.

We provide two cases for which P1 holds. First we show that, generically, for values of $v(N)$ which are slightly below v^E, P1 continues to hold. In other words, there is always a range of values of $v(N)$ for which the egalitarian solution is not is the core (hence some coalitions bind) and yet P1 holds.

Claim B. Assume that $\arg\max_{S \neq N} \frac{v(S)}{|S|}$ is a singleton (which holds whenever $\frac{v(S)}{|S|} \neq \frac{v(T)}{|T|}$ for all $S \neq T$). Then for ε small enough, P1 holds for all $v(N) \in (v^E - \varepsilon, v^E)$.

Intuitively, for values of $v(N)$ close to v^E, the core contains elements which are close to the egalitarian solution (which gives $\frac{v(N)}{n}$ to each player). Relaxing the constraints $x(S) \geq v(S)$ has thus little effect on the Nash product, while strengthening $x(N) \leq v(N)$ has a first order (and negative) effect on the Nash product.

Next, we define a class of problems for which P1 holds as soon as the core is non-empty.

Claim C. Assume that there is a key player i^*, that is, a player such that $v(S) = 0$ if $i^* \notin S$. Then condition P1 holds for all $v(N) \in (v_*, +\infty)$.

Intuitively, consider adding Δ to all constraints (thus strengthening the subcoalition constraints $x(S) \geq v(S)$ and weakening the grand team constraint $x(N)$). Giving that additional Δ to the key player results in an allocation that satisfies all the constraints, and that increases the Nash product.

4 Applications

In this Section we provide a characterization of the coalitional Nash bargaining solution, which in turn is used to provide a measure of the strength of coalitions in an asymptotically efficient equilibrium. We next illustrate this characterization through a series of applications.

4.1 The coalitional Nash bargaining solution: Another characterization

Consider a vector of non-negative weights $\mu = (\mu_S)_{S\in\mathcal{S}}$. We denote by $\mathcal{S}_i$ (respectively $\widetilde{\mathcal{S}_i}$) the set of coalitions in $\mathcal{S}$ to which i belongs (respectively *does not* belong), and we let

$$m_i^\mu = \sum_{S\in\tilde{\mathcal{S}}_i} \mu_S.$$

The following Proposition provides a characterization of the Coalitional Nash bargaining solution, which is obtained by considering the first-order conditions of the Lagrangian maximization associated with the maximization of the Nash product.

Proposition 3. *Let* $A1$ *and* $P1$ *hold. Then there exists a set of coalitions* $\mathcal{S}^* \subset \mathcal{S} \setminus N$*, an allocation* x^*, a vector of weights μ^* *and a scalar* $\alpha \in \{0,1\}$ *with the properties that*

$$\begin{aligned} x^* &\in \mathcal{C} & (2)\\ \forall S &\in \mathcal{S}^*,\ \mu_S > 0 \text{ and } x^*(S) = v(S) & (3)\\ \forall S &\notin \mathcal{S}^*,\ \mu_S = 0 & (4)\\ x^*(N) &= v(N) & (5)\\ (\alpha + m_i^\mu)x_i^* &= (\alpha + m_1^\mu)x_1^* \text{ for all } i. & (6) \end{aligned}$$

Besides, the vector x^* *is uniquely defined and coincides with the coalitional Nash bargaining solution.*

From the viewpoint of our non-cooperative game (to be analyzed later on), the set $\mathcal{S}^*$ will be interpreted as the set of credible coalitions, and μ_S will be interpreted as the strength of coalition S. Specifically, a coalition S is *credible* if it has a strictly positive strength $\mu_S > 0$. In the next Section, we will show that in the bargaining game, the set of credible coalitions coincides with the coalitions that are proposed with positive probability in equilibrium.[27] Also the strength of a coalition will be related to the probability that it is proposed in equilibrium. Note that, by condition (3), if S is credible then we must have $x^*(S) = v(S)$

We make a few observations about the coalitional Nash bargaining solution and the corresponding set of credible coalitions:

(i) Any credible coalition $S \neq N$ gets no surplus in addition to $v(S)$, i.e. $x^*(S) = v(S)$.

(ii) If the coalitional Nash bargaining solution x^* is such that there is *no binding blocking constraint*, i.e. $x^*(S) > v(S)$ for all $S \neq N, \varnothing$, then there is no credible (sub)coalition and the solution must coincide with the egalitarian solution x^E in which every agent i gets $v(N)/n$. Conversely, if the egalitarian solution x^E lies in the core, x^E is the coalitional Nash bargaining solution.

(iii) For generic values of $v(S)$, there are at most $N - 1$ coalitions $S \neq N$ that may be credible. Indeed, for each credible coalition S, one must have that $\sum_{i \in S} x_i = v(S)$. Given that $x(N) = v(N)$ holds and that there are only n unknowns x_1,...x_n, there can be at most $n - 1$ coalitions $S \neq N$ that can credibly form.

(iv) For generic values of $v(S)$, if a coalition S is not credible then $x^*(S) > v(S)$, and reducing slightly the value of $v(S)$ does not affect the solution, as the same (x^*, μ^*) solve the constraints and the solution is unique.

(v) Call $\mathcal{S}_i^*$ the set of credible coalitions to which i belongs. If $\mathcal{S}_i^* \supset \mathcal{S}_j^*$, then $x_i^* \geq x_j^*$. So in particular, if $\mathcal{S}_i^* = \mathcal{S}_j^*$, then $x_i^* = x_j^*$. This follows from (6), and it makes precise the intuition according to which the more credible coalitions a player belongs to, the stronger his bargaining position, and thus the higher his equilibrium payoff.

We now consider a series of examples for which we characterize the coalitional Nash bargaining solution.

4.2 The two player case.

Let $n = 2$, $v(N) = 1$ and $v(1) = w_1$, $v(2) = w_2$ with $w_1 + w_2 < 1$, and $w_1 \geq w_2$. There are two cases:

[27] That probability will tend to 0 as players get very patient. So the fact that coalitions other than N are proposed will not generate inefficiencies at the limit.

- So long as $w_2 \leq w_1 \leq \frac{1}{2}$, then $x_1^* = x_2^* = \frac{1}{2}$. This is an illustration of properties (ii) and (iv) above.

- If $w_1 \geq 1/2$, then $x_1^* = w_1$ and $x_2^* = 1 - w_1$.

The properties of the coalitional Nash bargaining solution in the two-player case are reminiscent of the literature on two-person bargaining with outside options (Binmore-Shaked-Sutton, 1989) in which the outside option of a player can be interpreted as the value of the coalition composed of this player only. As in the work on two-person bargaining with outside option, we find according to the characterization derived above that as long as $w_i \leq 1/2$, it is not credible to take the outside option, and the outcome remains equal to $(1/2, 1/2)$. Once a player may get $w_i \geq 1/2$ he must be getting at least that amount, however, in equilibrium he cannot get more than that amount, because otherwise, the outside option would cease to be credible.

4.3 A three player example.

Let $n = 3$, $v(N) = 1$, $v(12) = w_2$, $v(13) = w_3$. We assume that $w_3 \leq w_2 < 1$, and that all other coalitions yield 0. In that example, player 1 is a key player: he must be part of any coalition that generates a positive surplus. According to claim C we know that there exists an asymptotically efficient equilibrium as soon as A1 holds. We characterize below the coalitional Nash bargaining solution and the corresponding credible coalitions as a function of (w_2, w_3) by looking at all possible sets of credible coalitions and seeing what constraints on (w_2, w_3) apply.

- $\mathcal{S}^* = \emptyset$. Then $\mu_S = 0$ for all $S \neq N$, hence

  $$x_1^* = x_2^* = x_3^* = \frac{1}{3}.$$

  This is the solution so long as $w_3 \leq w_2 \leq \frac{2}{3}$.

- $\mathcal{S}^* = \{12\}$. Then $\mathcal{S}_1^* = \mathcal{S}_2^*$ so $x_1^* = x_2^*$. And $\mu_{12} > 0$, so $x_1^* + x_2^* = w_2$. It follows that

  $$x_1^* = x_2^* = \frac{w_2}{2} \text{ and } x_3^* = 1 - w_2$$

  This is the solution as long as $x^*(13) \geq w_3$, and $x_1^* \geq x_3^*$, that is,

  $$1 - \frac{w_2}{2} \geq w_3 \text{ and } w_2 \geq \frac{2}{3}$$

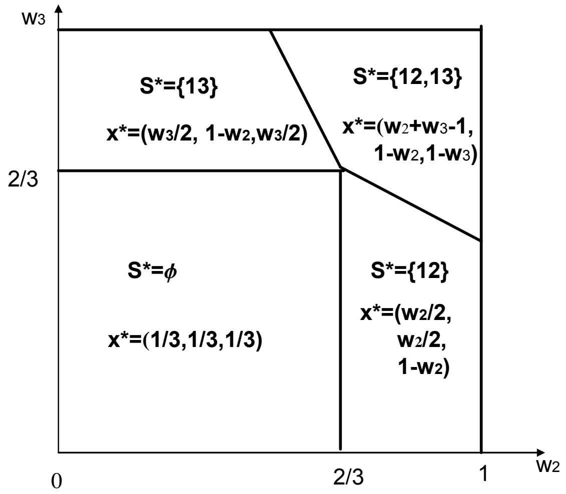


Figure 1:

- $\mathcal{S}^* = \{13\}$. This requires (see above, exchanging the role of 2 and 3) $1 - \frac{w_3}{2} \geq w_2$ and $w_3 \geq \frac{2}{3}$. Since $w_2 \geq w_3$, these inequalities are not compatible.

- $\mathcal{S}^* = \{12, 13\}$. Then $x_1^* + x_2^* = w_2$ and $x_1^* + x_3^* = w_3$ so

$$x_1^* = w_2 + w_3 - 1,\ x_2^* = 1 - w_3,\ x_3^* = 1 - w_2,$$

  and (6) requires $x_1^* \geq x_2^*$ and $x_1^* \geq x_3^*$, that is:

$$w_3 + \frac{w_2}{2} \geq 1.$$

These cases are summarized in Figure 1.

To sum up, so long as both coalitions 12 and 13 yield less than 2/3, they are not credible and thus w_2 and w_3 cannot affect the outcome of the bargaining game. When at least one coalition, say 12, may get at least 2/3, that coalition becomes credible. However in equilibrium, players 1 and 2 cannot jointly obtain more than w_2 (because if it were the case, the coalition would cease to be credible). Observe that it is not necessary that a coalition gets at least 2/3 to be credible. If $v(12)$ is large, 3 must be getting a small payoff, making the coalition 13 attractive to player 1. (See Figure 1).

It is interesting to contrast the results obtained above with the results obtained by Chatterjee et al. (1993) and Ray (2008) (chapter 7) in the employer-employee example in which the first rejector of a proposal is the next proposer. The employer-employee corresponds to the situation in which $w_2 = w_3 = 1 - a$ and a is very small. In our analysis, player 1 puts players 2 and 3 in competition and he is able to obtain a payoff of $1 - 2a$ whereas 2 and 3 get only a. By contrast, in the rejector-proposes protocol considered by Chatterjee et al and Ray, no matter who the first proposer is there must be inefficiencies (in the limit in which players are very patient).[28] This example illustrates that efficiency is more easily obtained in the random-proposer protocol than in the rejector-proposes protocol. We will come back to the comparison with this protocol in Section 6.

Our next example involves four players. In contrast to the previous examples, the coalitional Nash bargaining solution there is not solely determined by the set $\mathcal{S}^*$of credible coalitions.

4.4 A 4-player example.

We provide an example where, unlike the previous 2- and 3-player examples, finding the solution requires computing the coalition strength vector.

Let $v(N) = 1$, $v(12) = w_{12}$, $v(13) = w_{13}$. Other coalitions yield 0. Observe that whenever A1 holds, condition P1 trivially holds in this case as player 1 is a key player. We consider the case where both 12 and 13 would be credible coalitions. This occurs if and only if $w_{12} > 1/2$ and $w_{13} > 1/2$. The solution must satisfy:

$$\begin{aligned} 1 &= x_1^* + x_2^* + x_3^* + x_4^* \\ w_{12} &= x_1^* + x_2^* \\ w_{13} &= x_1^* + x_3^*. \end{aligned}$$

These constraints alone are not sufficient to derive the solution. Player 1 belongs to all credible coalitions, player 2 does not belong to 13, player 3 does not belong to 12, player 4 does not belong to either 12 or 13. (6) thus implies:

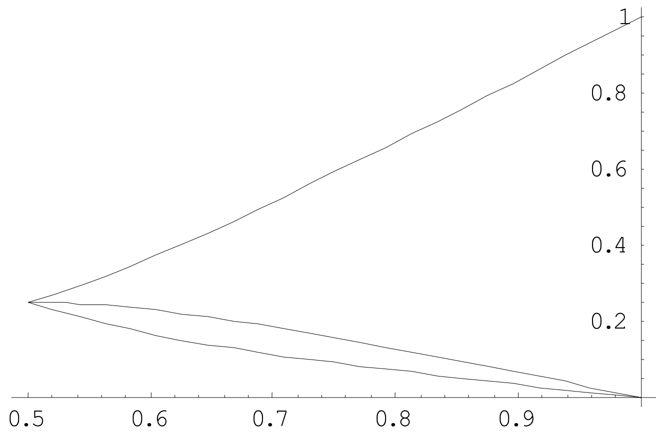
$$x_1^* = (1 + \mu_{13})x_2^* = (1 + \mu_{12})x_3^* = (1 + \mu_{12} + \mu_{13})x_4^*$$

Solving for μ_{12} and μ_{13} yields

$$\frac{1}{x_2^*} + \frac{1}{x_3^*} = \frac{1}{x_1^*} + \frac{1}{x_4^*}$$

[28] The reason for the inefficiency in the rejector-proposes protocol is that player 1 is unable to put players 2 and 3 in competition: indeed, when player 1 makes an offer to the grand coalition, each of players 2 and 3 can seize the opportunity to make a counter-offer if not happy with the offer, which in turn implies that each player must be getting at least $\frac{1-a}{2}$ to accept an offer, thereby leading players to prefer not approaching the grand coalition but a two-player coalition instead.

This relationship can then be used to derive the payoff profile x^*. For $w_{12} = w_{13} = w > 1/2$, we plot in Figure 2 the values of $x_1^* > x_2^* = x_3^* > x_4^*$ as a function of w.



4.5 Convex games.

Some of the literature on coalition formation has considered convex games (see Chatterjee et al. (1993)). A game is *strictly convex* if the following property (PSC) holds:

> *PSC: For all S, S' such that $S \cap S' \subsetneq S'$ and $S \cap S' \subsetneq S$, $v(S \cup S') > v(S) + v(S') - v(S \cap S')$.*

It is well known that strictly convex games have a non-empty interior core. We have:

> **Claim D**: *When PSC holds, property P1 holds and the set of credible coalitions $\mathcal{S}^* = \{S^{(1)}, ...S^{(m)}\}$ must be nested, that is, $S^{(1)} \subset S^{(2)} ... \subset S^{(m)} \subset N$.*

This claim implies that strictly convex games have an asymptotically efficient equilibrium (by Proposition 1) and that the profile of payoffs obtained in the limit coincides with the coalitional Nash bargaining solution (by Proposition 2).

The details of the proof appear in the appendix. The key observation (step 1 in the proof) is that if two coalitions S' and S'' are binding in the core allocation x the sense that $x(S') = v(S')$ and $x(S'') = v(S'')$ then either $S' \subseteq S''$ or $S'' \subseteq S'$. This is because otherwise due to the strict convexity either $S' \cap S''$ or $S' \cup S''$ would block x. This fact implies that in strictly convex games, there must exist at least one player who belongs to all credible coalitions, thereby guaranteeing that property P1 holds, and it also implies that the set of binding coalitions in the maximization of the Nash product must be nested.

Note that from Claim D, and from the observation that players who belong to the same set of credible coalitions must be getting the same payoff, one can deduce an algorithm for finding out the coalitional Nash bargaining solution of strictly convex games. Such a construction appears in Dutta and Ray (1989) and is detailed in Compte and Jehiel (2008).

4.6 Buyer-submodular games.

The recent literature on multi-object auctions (see Ausubel and Milgrom, 2002) has considered exchange problems between one seller and several buyers, and it has emphasized the role of the buyer-submodularity property (to be defined next) that ensures that the pivotal mechanism allocation (in which each buyer gets a payoff equal to the surplus he generates) lies in the core.

Formally, we assume that player 1 is the seller and players $i \neq 1$ are the buyers, so we have $v(S) > 0$ if and only if $1 \in S$ (which immediately implies by claim C that P1 holds as soon as the core has a non-empty interior). The function $v(.)$ is (strict) buyer-submodular if the following property holds:

PBS: For any $S \subsetneq N$ and $i \in S$, with $i \neq 1$, $v(S) - v(S \setminus \{i\}) > v(N) - v(N \setminus \{i\})$.

We have:

Claim E: *When PBS holds, property P1 holds and credible coalitions can only be of the form* $N \setminus \{i\}$, $i \neq 1$.

We provide a direct proof of this claim in the appendix. This claim is not surprising as we know from Ausubel and Milgrom (2002) that for buyer-submodular games, the set of core allocations coincides with:

$$\bar{C}(v) \equiv \{x,\ x(N) = v(N) \text{ and } x_i \leq v(N) - v(N \setminus \{i\}) \text{ for all } i \neq 1\}$$

Finding the coalitional Nash bargaining solution thus reduces to finding a profile x satisfying

$$\begin{aligned} &\underset{x_2,\ldots x_n}{Max}\,(v(N) - \sum_{i \neq 1} x_i) \prod_{i \neq 1} x_i \\ x_i \quad &\leq \quad v(N) - v(N \setminus \{i\}) \text{ for all } i \neq 1 \end{aligned}$$

When all constraints are binding for the coalitional Nash bargaining solution x^*, then $x_i^* = v(N) - v(N \setminus \{i\})$ for all $i \neq 1$, and the solution coincides with the pivotal mechanism allocation,[29] which is the core allocation most preferred by the buyers. In general, the coalitional Nash bargaining solution need not coincide with the pivotal allocation. This is because in our bargaining protocol all players have the same chance of being the proposer, which equalizes the bargaining power of the seller and the buyers.

5 Analysis of the bargaining game.

We prove Propositions 1 and 2. Throughout the proof, for a given equilibrium, we denote by $q^{i,S}$ the probability that player i makes a proposal to S, and by $x^{i,S}$ the vector of payments that player i proposes to S, and by $u = (u_1, ..., u_n)$ the equilibrium value. We let $\underline{u}_i = \delta u_i$, and $\Delta = v(N) - \underline{u}(N)$. $\underline{u}_i$ corresponds to the threshold offer that player i would accept, and Δ to the equilibrium inefficiency induced by a rejection at the current date.[30] We let $\pi_i = \frac{1}{n} \sum_{j,S \in \tilde{\mathcal{S}}_i} q^{j,S}$. The probability π_i corresponds to the equilibrium probability that player i is not part of a coalition that forms (at the current date). We also let $\bar{u}_i$ denote the payoff that player i obtains in equilibrium when he is selected to make the proposal. Finally, we will refer to (P_Δ) as the program:

$$\arg \max_{x \in \mathcal{C}(\Delta)} \prod_{i \in N} x_i$$

and to x^*_Δ as its (unique solution).[31]

We start with Proposition 2.

Step 1: In equilibrium, if player i proposes to coalition S, it must be that he offers $\underline{u}_j = \delta u_j$ to player j, who accepts that offer. We note that since $v(N) > v(S)$ for all $S \neq N$, we must have $\underline{u}(N) \leq \delta v(N) < v(N)$. This implies that in equilibrium, when selected to move, players make proposals that are accepted with probability 1.

Step 2: There exists Δ_0 such that if $\Delta \leq \Delta_0$, then all players propose to the coalition N with positive probability.

Indeed, when $A1$ holds, a proposal to a coalition $S \neq N$ (which would be accepted in equilibrium by step 1) would generate an inefficiency bounded away from 0, say η. So if a player were to propose N with probability 0, $u(N)$ would be bounded away from $v(N)$ (by η/n). Hence, if $\Delta \leq \eta/n$, it cannot be that a player proposes N with 0 probability.

[29] This is the allocation in which each player $i \neq 1$ gets a payoff equal to the surplus he generates, that is, $v(N) - v(N \setminus \{i\})$.

[30] That is, as compared with the situation in which the grand team forms instantaneously.

[31] The solution is unique because $\mathcal{C}(\Delta)$ is convex and because the sets $\{x, \prod_{i \in N} x_i \geq \beta\}$ are strictly convex.

Step 3: By step 2, in the event player i proposes, he gets a payoff $\bar{u}_i$ that satisfies $\bar{u}_i - \underline{u}_i = v(N) - \underline{u}(N)$. Besides, if $q^{i,S} > 0$, then $v(N) - \underline{u}(N) = v(S) - \underline{u}(S)$. By definition of Δ, we thus have that for all coalitions S that are proposed in equilibrium:

$$\underline{u}(S) = v(S) - \Delta.$$

Finally, if $i \in S$ and $q^{i,S} = 0$, then it must be $v(S) - \underline{u}(S) \leq v(N) - \underline{u}(N)$, as otherwise i would strictly prefer proposing to S.

It follows that for all S,

$$\underline{u}(S) \geq v(S) - \Delta.$$

Hence $\underline{u} \in \mathcal{C}(\Delta)$

Step 4: Equilibrium conditions can be written as follows:

$$u_i = \frac{1}{n}\bar{u}_i + \pi_i * 0 + (1 - \frac{1}{n} - \pi_i)\underline{u}_i$$

which, since $u_i = \frac{\underline{u}_i}{\delta}$ and $\bar{u}_i - \underline{u}_i = \Delta$, can re-written as:

$$(\frac{1-\delta}{\delta} + \pi_i)\underline{u}_i = \frac{1}{n}\Delta$$

Step 5: We must have $\underline{u} = x^*_\Delta$ Besides, for any $\Delta' < \Delta$, $\mathcal{N}(\Delta') > \mathcal{N}(\Delta)$.
This step results from the following Lemma, which we prove in the Appendix.

Lemma: *For any* $\Delta \geq 0$, let H_Δ denote the set of triplets (x, λ, η) *where* $\lambda = (\lambda_S)_S$ *with all* λ_S *non-negative and* $\eta \in \Re$ such that for some $a > 0$, the *following properties hold:*

$$x \in \mathcal{C}(\Delta) \quad (7)$$

$$\lambda_S x(S) = \lambda_S(v(S) - \Delta)) \text{ for all } S \quad (8)$$

$$(\lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S)x_i = a, \quad (9)$$

$$a = \prod_i x_i \quad (10)$$

$$\gamma = \lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S \quad (11)$$

Consider $(x, \lambda, \eta) \in H_\Delta$. *Then* $x = x^*_\Delta$. *Besides, if* $\gamma > 0$, then for any $\Delta' < \Delta$, $\mathcal{N}(\Delta') > \mathcal{N}(\Delta)$. And if $\gamma \leq 0$, then for any $\Delta' \geq \Delta$, $\mathcal{N}(\Delta') \geq \mathcal{N}(\Delta)$.

Equilibrium conditions derived in step 4 precisely allow us to find $\lambda, \gamma > 0$ such that $(\underline{u}, \lambda, \gamma) \in H_\Delta$ (thereby showing step 5). Namely, set $\lambda_S = \frac{\delta}{1-\delta}\frac{1}{n}\sum_{j\in N} q^{j,S}$ and $\lambda_N = 1+\sum_S \lambda_S$. So it is immediate that $\gamma > 0$. Note that $\sum_{S\in\tilde{\mathcal{S}}_i} \lambda_S = \frac{\delta}{1-\delta}\pi_i$. So by construction,

$$(\lambda_N - \sum_{S\in\mathcal{S}_i} \lambda_S)\underline{u}_i = (1 + \frac{\delta}{1-\delta}\pi_i)\underline{u}_i = \frac{\delta}{1-\delta}\frac{1}{n}\Delta.$$

So (9) holds with $a = \frac{\delta}{1-\delta}\frac{1}{n}\Delta$. Multiplying all λ_S and η by the same constant, we can also ensure that (10). Besides, for any S, either $q^{i,S} = 0$ for all i, in which case $\lambda_S = 0$, or there exists i such that $q^{i,S} > 0$, in which case $\underline{u}(S) = v(S) - \Delta$. So (8) holds. Finally, step 3 implies that $\underline{u} \in \mathcal{C}(\Delta)$. This concludes step 5.

Final Step. If one considers a sequence of discount factors δ^k tending to 1 and associated equilibrium values $u^{(k)}$ converging to an efficient allocation u^*, then there must exists a sequence $\Delta^{(k)}$ tending to 0 such that $\delta^{(k)}u^{(k)} = x^*_{\Delta^{(k)}}$, hence u^* must coincide with x^*, the coalitional Nash bargaining solution. Besides, since $\mathcal{N}(.)$ is continuous in Δ, and since $\mathcal{N}(\Delta^{(k)})$ is a strictly increasing sequence, Property $P1$ must hold.[32]

We now turn to Proposition 1. Note that Proposition 2 already implies that if property $P1$ does not hold then equilibrium outcomes with patient players must remain bounded away from efficiency. We now show that if $P1$ holds, then for patient enough players, we can construct equilibria with values arbitrarily close to the coalitional Nash bargaining solution.

Consider any $(x, \lambda, \gamma) \in H_\Delta$. By construction x corresponds to the solution of (P_Δ) and it is uniquely defined: $x = x^*_\Delta$. Since property $P1$ holds, there exists Δ_0 such that $\mathcal{N}(.)$ is strictly decreasing on $(0, \Delta_0)$, thus for any $\Delta < \Delta_0$ and $(x, \lambda, \gamma) \in H_\Delta$, we must have $\gamma > 0$. Also note that since $\mathcal{C}(\Delta)$ has a non-empty interior for Δ close enough to 0, all $x_i's$ are bounded away from 0, λ_N must be bounded (for each given Δ) and thus so are all the λ_S (since $\gamma > 0$).

For each $S \in \mathcal{S}$, either $\lambda_S = 0$, and we set $q^{i,S} = 0$ for all i; or $\lambda_S > 0$, and we set $q^{i,S} = \frac{n}{|S|}\frac{1-\delta}{\delta}\frac{\lambda_S}{\gamma}$ if $i \in S$.[33] Next define $\pi_i = \frac{1}{n}\sum_{j,S\in\tilde{\mathcal{S}}_i} q^S_j$. We have:

$$\pi_i = \frac{1-\delta}{\delta}\sum_{S\in\tilde{\mathcal{S}}_i}\sum_{j\in S}\frac{1}{n}q^{j,S} = \frac{1-\delta}{\delta}\sum_{S\in\tilde{\mathcal{S}}_i}\frac{\lambda_S}{\gamma}$$

[32] Indeed, assume by contradiction that P1 does not hold. Then there must exist a sequence $\Delta_m \searrow 0$ such that $\mathcal{N}(\Delta_m) \geq \mathcal{N}(\Delta_{m+1})$. Since $\mathcal{N}(.)$ is continuous, we must have $\mathcal{N}(\Delta_m) \searrow \mathcal{N}(0)$. Now observe that for any $\Delta \in (0, \Delta_1)$, we must have $\mathcal{N}(\Delta) \geq \min(\mathcal{N}(\Delta_1), \mathcal{N}(0)) = \mathcal{N}(0)$. [This is because if $x^1 \in \mathcal{C}(\Delta_1)$ and $x^0 \in \mathcal{C}(0)$, then $y \equiv \alpha x^1 + (1-\alpha)x^0 \in \mathcal{C}(\alpha\Delta_1)$, and because $\prod_i y_i \geq \min\{\prod_i x^0_i, \prod_i x^1_i\}$]. So there could not exit a sequence $\Delta^{(k)} \searrow 0$ such that $\mathcal{N}(\Delta^{(k)}) \nearrow \mathcal{N}(0)$. Contradiction.

[33] Observe that $\sum_{S\neq N} q^{i,S}$ add up to less than 1 for δ sufficiently close to 1.

Thus

$$(\frac{1-\delta}{\delta}+\pi_i)x_i = \frac{1-\delta}{\delta}\frac{1}{\gamma}(\gamma + \sum_{S\in\widetilde{\mathcal{S}}_i}\lambda_S)x_i = \frac{1-\delta}{\delta}\frac{1}{\gamma}(\lambda - \sum_{S\in\mathcal{S}_i}\lambda_S)x_i = \frac{1-\delta}{\delta}\frac{\mathcal{N}(\Delta)}{\gamma}.$$

Our aim will be to find Δ and $(x,\lambda,\gamma)\in H_\Delta$ such that

$$\frac{1-\delta}{\delta}\frac{\mathcal{N}(\Delta)}{\gamma} = \frac{1}{n}\Delta \tag{12}$$

For now, assume that we can find such a Δ. We consider the strategies where each player i chooses S with probability $q^{i,S}$, and threshold values are set to $\underline{u}_i = x_i$. We check below that these strategies constitute an equilibrium. We do that using the one-shot deviation principle. Assume that other players behave in this way, and that from next date on, player i also behaves in this way. We show that it is then optimal for player i to behave in this way today.

Indeed, by construction, for any i and S such that $q^{i,S}>0$, $v(N)-x(N-\{i\}) = \Delta + x_i = v(S)-x(S-\{i\})$, so player i is indeed indifferent between choosing S or N. And for any i,S such that $q^{i,S}=0$, $v(N)-x(N)=\Delta \geq v(S)-x(S)$, so not choosing S is optimal. Finally, computed from next date, player i's expected payoff is:

$$u_i = \frac{1}{n}(v(N)-x(N-\{i\}) + (1-\frac{1}{n}-\pi_i)x_i.$$

It is thus optimal for player i to accept offers above δu_i. By construction we have:

$$u_i - \frac{x_i}{\delta} = \frac{1}{n}(v(N)-x(N)) - (\frac{1-\delta}{\delta}+\pi_i)x_i = \frac{1}{n}(v(N)-x(N)) - \frac{\Delta}{n} = 0$$

It is thus optimal for player i to follow the proposed strategy.

It remains to show that one can find Δ that solves (12), and that such Δ tends to 0 when δ tends to 1.

For any $\Delta>0$ and any $(x,\lambda,\gamma)\in H_\Delta$ let $h(\delta,x,\lambda,\gamma) = \min(\frac{n(1-\delta)\mathcal{N}(\Delta)}{\delta\gamma},\Delta_0)$. Fix δ_0 small enough so that for all $(x,\lambda,\gamma)\in H_{\Delta_0}$ and $\delta\geq\delta_0$, $h(\delta,x,\lambda,\gamma)<\Delta_0$. Now fix any $\delta\geq\delta_0$, and choose $\underline{\Delta}$ small enough so that for all $(x,\lambda,\gamma)\in H_{\Delta_0}$, $h(\delta,x,\lambda,\gamma)>\underline{\Delta}$. The correspondence $\Delta\to h(H_\Delta,\delta)$ is non-empty convex, defined from $[\underline{\Delta},\Delta_0]$ into itself, and it has a closed graph. Applying Kakutani fixed point theorem, it follows that there exists Δ^* and $(x,\lambda,\gamma)\in H_{\Delta^*}$ such that $h(\delta,x,\lambda,\gamma)=\Delta^*$. Since by construction Δ^* must be different from Δ_0, $h(\delta,x,\lambda,\gamma)<\Delta_0$, hence $h(\delta,x,\lambda,\gamma) = \frac{n(1-\delta)\mathcal{N}(\Delta^*)}{\delta\gamma} = \Delta^*$. So for any $\delta\geq\delta_0$, we can find $\Delta^*(\delta)$ and $(x,\lambda,\gamma)\in H_{\Delta^*}$ that solves (12), as desired.

Finally, consider any $\Delta \in (0, \Delta_0)$. For δ close enough to 1, it must be the case that $\frac{n(1-\delta)\mathcal{N}(\Delta')}{\delta\gamma} < \Delta'$ for all $\Delta' \geq \Delta$. So for δ small enough, $\Delta^*(\delta)$ must be smaller than Δ. Since this is true for Δ arbitrarily small, $\Delta^*(\delta)$ must get close to 0 as δ tends to 1.

6 Discussion.

The role of $P1$.

It is instructive to revisit a situation in which $P1$ does not hold to better understand why an efficient stationary equilibrium fails to exist in such a case.

To that end, consider the situation of claim A in which the core is reduced to payoff vectors in which at least one player gets a very small payoff, and there is a gap between the value of the grand coalition $v(N)$ and the value of any other coalition $v(S)$, $S \neq N$. Call $\eta \equiv \min_{S \subsetneq N}(v(N) - v(S))$ this gap.

We will explain below that in equilibrium, as δ gets close to 1, any player's payoff must remain bounded away from 0. This in turn will imply that asymptotically, the equilibrium value vector cannot be a core allocation, hence it cannot be efficient (by Proposition 0).

As explained in step 4 of the general proof, in any stationary equilibrium, one must have:

$$(\frac{1-\delta}{\delta} + \pi_i)\underline{u}_i = \frac{1}{n}(\bar{u}_i - \underline{u}_i) \tag{13}$$

where $\bar{u}_i$ is the payoff that player i obtains in the event he is selected to make a proposal, $\underline{u}_i$ is the acceptance threshold of player i and π_i is the equilibrium probability that player i is not part of a coalition that forms in any given period. So one might expect to reduce $\underline{u}_i$ by having π_i large compared to $\frac{1-\delta}{\delta}$, that is, by increasing the probability that player i is not in the winning coalition. However, a larger value of π_i means a larger efficiency loss,[34] hence in turn a larger value of $\bar{u}_i - \underline{u}_i$, as the following sequence of inequalities show:

$$\bar{u}_i - \underline{u}_i \geq v(N) - \underline{u}(N) \geq v(N) - \delta(v(N) - \pi_i\eta),$$

where the first inequality follows because player i can always make a proposal to the grand coalition and extract the surplus $v(N) - \underline{u}(N)$ (in addition to $\underline{u}_i$) for himself and the second inequality follows because when a coalition in which i does not belong forms (which happens with probability π_i) there is an efficiency loss at least equal to η (remember that $\underline{u}(N)/\delta$ is the expected welfare obtained by the set of all players).

[34] This is because when a coalition in which i does not belong forms (which happens with probability π_i) there is an efficiency loss at least equal to η.

This inequality combined with equality (13) implies that:

$$u_i = \frac{\underline{u}_i}{\delta} \geq \frac{1}{n}\frac{(1-\delta)v(N)+\delta\pi_i\eta}{(1-\delta)+\delta\pi_i} \geq \frac{\eta}{n},$$

hence the lower bound on player i's equilibrium payoff.

Intuitively, we can expect to reduce u_i by increasing the probability that player i is not in the winning coalition, but increasing that probability generates inefficiency losses that player i himself can extract once he gets to be the proposer.

Inefficiencies in the rejector-proposes protocol.

Consider the following bargaining protocol. In period 1, any player i may be selected to be the proposer with probability $1/n$. A proposer in any period may choose a coalition S to which he makes an offer. Players respond sequentially if they accept the offer (in increasing order of their label). The first player who rejects the offer becomes the next proposer. If all players accept, the game ends. This game is similar to the one considered in Chatterjee et al. (1993) except that once a coalition forms it is the end (and the first proposer is chosen at random). We have:

> **Proposition 4**. *Let A1 hold. If the egalitarian solution does not lie in the core (i.e. if $v(N) < v^E$) then there does not exist an asymptotically efficient equilibrium in the rejector-proposes protocol with random choice of first proposer.*

Proof of Proposition 4. Call $x_i(\delta)$ the equilibrium payoff when i starts and the discount factor is δ. For an asymptotically efficient equilibrium to exist, it should be that $N \in \arg\max_S [v(S) - \delta x(S)]$. Indeed, otherwise, if, say, $S^* \in \arg\max_S [v(S) - \delta x(S)]$ then any $i \in S^*$ would strictly prefer making an offer to S^* rather than to N. Since any player $i \in S^*$ has a probability $\frac{1}{n}$ of being the first proposer, we would get significant - i.e. bounded away from 0 - inefficiencies.

This implies that for all i,

$$x_i(\delta) = v(N) - \delta \sum_{j \neq i} x_j(\delta)$$

which in turn imply that all $x_i(\delta)$ are equal and satisfy:

$$x_i(\delta) = \frac{1}{1-\delta+\delta n} v(N).$$

Thus $x(\delta)$ must be converging to the egalitarian solution. Since an asymptotically efficient equilibrium must lie in the core (the proof of this is similar to that in Proposition 0 and it is omitted here), we get the desired conclusion. **Q. E. D.**

Proposition 4 is analogous to the result in Chatterjee et al (1993) and Ray (2008) that for efficiency to obtain for *all* rejector-proposes protocols the egalitarian solution should lie in the core (they call this a situation in which the game is dominated by its grand coalition). Proposition 4 shows that this insight carries over to our framework in which only one coalition can form.[35] Comparing Proposition 4 with Proposition 1 illustrates a precise sense in which the random offer protocol induces a more efficient outcome than the rejector-proposes protocol.[36]

The coalitional Nash bargaining solution and the Shapley value.

The Shapley value is defined as the allocation x^{sh} satisfying:[37]

$$x_i^{sh} = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left[v(S \cup \{i\}) - v(S) \right] \tag{14}$$

From (14), it is readily verified that the value $v(S)$ of all coalitions S affect x_i^{sh} in the sense that x_i^{sh} is modified by a slight perturbation of any $v(S)$. Consider now the coalitional Nash bargaining solution and the properties highlighted in Section 4. A slight perturbation of $v(S)$ when S is not credible has no effect on the coalitional Nash bargaining solution (this is observation (iv) in Section 4) and there are at most $n-1$ credible coalitions other than N (this is observation (iii)). It follows that the Shapley value and the coalitional Nash bargaining solution cannot coincide in general.

Some papers (see in particular, Hart and Mas-Colell (1996)) have provided some non-cooperative foundation for the Shapley value. In light of our insights that at most N coalitions can credibly form in equilibrium, it may seem surprising that some non-cooperative bargaining models might give rise to the Shapley value. As it turns out, these models all make an assumption that forces all coalitions to play a role in equilibrium. This is done by breaking the stationarity assumption made in our model. Specifically, in Hart and Mas-Colell the

[35] The proof is in fact the same, despite the difference in the game structure.

[36] Chatterjee et al (1993) and Ray (2008) proceed from the analog of Proposition 4 to analyze when it is the case that for some well chosen first proposer asymptotic efficiency can be obtained in the rejector-proposes protocol (in which further coalitions can form after some players have left the game). They observe that in the employer-employee game (see above) this is impossible while for strictly convex games there is always such a protocol. It is an open question how the conditions for the existence of an asymptotically efficient equilibrium for *some* rejector-proposes protocol compare with condition P1 (the condition for the existence of an asymtotically efficient equilibrium in the random proposer protocol).

[37] Observe that the Shapley value is defined whether or not the core is empty. By contrast, the coalitional Nash bargaining solution can only be defined if the core is non-empty.

non-stationarity takes the form that with some positive (yet small) probability a proposer whose offer is rejected can no longer take part in the negotiation and receives 0 utility. By this assumption, all coalitions must play a role in equilibrium: in case of a rejection, there is always a chance that the other players have disappeared from the bargaining table, so the values of all coalitions must in turn affect the value to rejecting an offer, hence also the offers that are made in equilibrium.[38]

The coalitional Nash bargaining solution and the nucleolus.

Consider now the following approaches proposed in the cooperative game theory literature: the (pre-)kernel (Davis and Maschler, 1965) and the (pre-)nucleolus (Schmeidler, 1969). The nucleolus, which is a selection from the kernel is defined as follows. Let $e(S,x) = v(S) - x(S)$ denote the excess of coalition given the allocation x.[39] There are 2^n coalitions S, which for each allocation x can be ranked by non-increasing order of excess $e(S,x)$, i.e. $e(S_1,x) \geq e(S_2,x) ... \geq e(S_{2^n},x)$. The nucleolus denoted x^{nl} is a feasible allocation x which minimizes the vector $(e(S,x))_S$ according to the lexicographic order. Several remarks about the nucleolus are in order.

(a) When the core has a non-empty interior, the nucleolus x^{nl} lies in the core and it is such that the excess of every coalition $S \neq N, \varnothing$ is strictly negative, $e(S,x^{nl}) < 0$. Accordingly, in such a case the nucleolus is a core allocation such that there is no binding blocking constraint.[40]

(b) There are always exactly n subcoalitions $S \neq N, \varnothing$ whose value $v(S)$ matter for the determination of the nucleolus (see Maschler et al., 1972).

Thus, unlike the Shapley value, the nucleolus is not affected by the value of all subcoalitions S. Despite this common property with the coalitional Nash bargaining solution, we now highlight a few notable differences between these two solutions.

First, the nucleolus always has the property that there is no binding blocking constraint

[38] In Gul (1989) as in many papers of coalition formation, it is never possible to undo a coalition, as once a coalition forms it is assumed that the coalition reduces to a single player. This creates non-stationarities and again it forces all coalitions to be possibly forming in equilibrium.

[39] An allocation x is feasible if $e(N,x) \geq 0$ and coalition S is not blocking x if $e(S,x) \leq 0$.

[40] Any core allocation x is such that $\max_S e(S,x) = 0$ whereas a feasible allocation y outside the core would satisfy $\max_S e(S,y) > 0$, thereby implying that the nucleolus is in the core when the core is non-empty. When the core has a non-empty interior, it is readily verified that there is a core allocation x such that $e(S,x) < 0$ for all $S \neq N, \varnothing$. Any such allocation dominates according to the lexicographic order any core allocation y for which $e(S,y) = 0$ for some $S \neq N, \varnothing$ (any core allocation must satisfy $e(N,x) = e(\varnothing,x) = 0$).

(see remark (a)). In contrast, the coalitional Nash bargaining solution has this same property if and only if the egalitarian solution x^E lies in the core (see property ii in Section 4).[41] Second, there are always n subcoalitions S whose value affects the nucleolus whereas there can be at most $n-1$ (and possibly fewer) coalitions that affect the coalitional Nash bargaining solution.

The kernel and the nucleolus have been characterized in terms of basic axioms in the cooperative game theory literature (see, for example, Driessen, 1991). The basic axiom that is failed by the coalitional Nash bargaining solution (and that is met by the nucleolus and the kernel to which the nucleolus belongs) is the property referred to as RISE for "Relative Invariance to Strategic Equivalence" requiring that if v is replaced by $\alpha v+\beta$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^n$ where $(\alpha v+\beta)(S) := \alpha v(S) + \sum_{j\in S} \beta_j$ then the solution x^* should be replaced by $\alpha x^* + \beta$ (i.e. player i should get $\alpha x_i^* + \beta_i$).[42]

We note that other properties, in particular the reduced game property are met both by the nucleolus and the coalitional Nash bargaining solution. The reduced game property as considered in Maschler (1965) applies here. That is, let us define the reduction of v to coalition T with respect to x, (T, v^x), according to: $v^x(\varnothing) = 0$, $v^x(T) = v(N) - x(N\setminus T)$ and for $S \subset T$, $S \neq T$

$$v^x(S) = \max\left[v(S\cup R - x(R) \mid R \subseteq N\setminus T\right]$$

The coalitional Nash bargaining solution x^* has the reduced game property because for any $T \subset N$, $x_T^* = (x_i^*)_{i\in T}$ is the coalitional Nash bargaining solution of the game defined over T by (T, v^{x^*}).[43]

7 Conclusion

For coalition games with transferable utilities and non-empty cores, we have introduced a new solution, the coalitional Nash bargaining solution which is the vector of payoffs that

[41] If x^E does not lie in the core, there is always at least one credible coalition (for which $x^*(S) = v(S)$). It follows that: 1) Either x^E does not lie in the core, and then the nucleolus and the coalitional Nash bargaining solution must differ; 2) Or x^E lies in the core, and then the coalitional Nash bargaining coincides with x^E. Yet, in the latter case, for generic values of $v(S)$, the nucleolus differs from x^E.

To illustrate the latter point with two players, assume $v(N) = 1$, $v(\{1\}) = w_1 < 1/2$ and $v(\{2\}) = w_2 < 1/2$. The egalitarian solution is $x^E = (1/2, 1/2)$, it lies in the core, and the nucleolus is given by $x^{nl} = (\frac{1+w_1-w_2}{2}, \frac{1+w_2-w_1}{2})$.

[42] RISE implies in the two-player case $n = 2$, $v(N) = 1$ and $v(1) = w_1$, $v(2) = w_2$ that i should get $\frac{1+w_i-w_j}{2}$, which does not coincide with the coalitional Nash bargaining solution, see Section 3.

[43] This is because it just amounts to optimizing over x_T, taking $x_{N\setminus T} \equiv x^*_{N\setminus T}$ as given.

maximizes the product of all players' payoffs subject to the core constraints (i.e. such that every group S should get at least its value $v(S)$). This solution was shown to coincide with the asymptotically efficient equilibrium of a non-cooperative game in which one coalition can form and all players have an equal chance of being the proposer in every period, as long as the core is non-empty and some extra condition referred to as P1 is met. In future work, we plan to revisit a number of multi-agent bargaining or contracting applications with the coalitional Nash bargaining solution.

References

[1] Ausubel, L. and P. Milgrom (2002): 'Ascending auctions with package bidding,' *Frontiers of Theoretical Economics*, art. 1.

[2] Baron, D. and Ferejohn (1989), "Bargaining in Legislatures", *American Political Science Review*, **90**, 316-330.

[3] Binmore, K., A. Rubinstein and A. Wolinsky (1986): "The Nash Bargaining Solution", *RAND Journal of Economics*, **17**, 176-188.

[4] Binmore, K., A. Shaked, and J. Sutton (1989): 'An outside option experiment,' *Quarterly journal of economics*, **104**, 753-770.

[5] Bloch, F. (1996): 'Sequential formation of coalitions in games with externalities and fixed payoff division,' *Games and Economic Behavior*, **14**, 90-123.

[6] Chatterjee, K., B. Dutta; D. Ray and K. Sengupta (1993): 'A Noncooperative Theory of Coalitional Bargaining,' *Review of Economic Studies*, **60**, 463-47.

[7] Compte, O. and P. Jehiel (2008): "The Coalitional Nash Bargaining Solution", working paper.

[8] Davis, M. and M. Maschler (1965): 'The kernel of a cooperative game,' *Naval Research Logistic Quarterly*, **12**, 223-259.

[9] Driessen, T.S.H (1991): 'A Survey of Consistency Properties in Cooperative Game Theory,' *SIAM Review*, **33**, 43-59.

[10] Dutta, B. and D. Ray (1989): 'A Concept of Egalitarianism under Participation Constraints,' *Econometrica*, **57**, 615-635.

[11] Gomes, A. and P. Jehiel (2005): 'Dynamic Processes of Social and Economic Interactions: On the Persistence of Inefficiencies,' *Journal of Political Economy*, **113**, 626-667

[12] Gul, F. (1989): 'Bargaining Foundations of Shapely Value,' *Econometrica*, **57**, 81-95.

[13] Hart, S. and A. Mas-Colell (1996): 'Bargaining and Value,' *Econometrica* **64**, 357-380

[14] Konishi H. and D. Ray (2003): 'Coalition Formation as a Dynamic Process,' *Journal of Economic Theory* **110**, 1-41.

[15] Krishna, V. and R. Serrano (1996): 'Multilateral Bargaining,' *Review of Economic Studies*, **63**, 61-80.

[16] Maschler, M., B. Peleg and L. S. Shapley (1979): 'Geometric Properties of the kernel, nucleolus, and related solution concepts,' *Mathematics of Operation Research*, **4**, 303-338.

[17] Milgrom, P. (2007): 'Package Auctions and Package Exchanges', *Econometrica*, **75**, 935-966.

[18] Moldovanu, B and E. Winter (1995): 'Order Independent Equilibria,' *Games and Economic Behavior*, **95**, 21-35.

[19] Okada, A. (1996): 'A Noncooperative Coalitional Bargaining Game with Random Proposers,' *Games and Economic Behavior*, **16**, 97-108.

[20] Okada, A. (2007): 'Coalitional Bargaining Games with Random Proposers: Theory and Application,' mimeo.

[21] Perry, M. and P. Reny (1994): 'A Noncooperative View of Coalition Formation and the Core,' *Econometrica*, **62**, 795-817.

[22] Ray, D. (2007): *A Game Theoretic Perspective on Coalition Formation,* The Lipsey Lectures, Oxford University Press.

[23] Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model", *Econometrica* **50**, 97-109.

[24] Ray, D. and R. Vohra (1999): "A Theory of Endogenous Coalition Structure ", *Games and Economic Behavior*, 26, 286–336.

[25] Selten, R. (1981): 'A non-cooperative model of characteristic function bargaining,' in Bohm and Nachtkamp (eds).

[26] Schmeidler, D. (1969): 'The Nucleolus of a Characteristic Function Game,' *SIAM Journal of Applied Mathematics*, **17**, 1163-1170.

[27] Shapley, L. (1953): 'A value for n-person games', *Annals of Mathematical Studies*, **28**, 307-217.

8 Appendix

Recall that we refer to (P_Δ) as the program:

$$\arg\max_{x \in \mathcal{C}(\Delta)} \prod_{i \in N} x_i$$

and to x^*_Δ as its (unique) solution. Since $\mathcal{C}(\Delta)$ has non-empty interior, the Kuhn and Tucker Theorem applies (see for example Theorem 28.3. in Rockafellar 1972). Define

$$L(x,\lambda,\Delta) = \prod_{i \in N} x_i + \sum \lambda_S(x(S) - (v(S) - \Delta)) + \lambda_N(v(N) - \Delta - x(N)).$$

A vector x is the solution to (P_Δ) if and only if there exists $\lambda \geq 0$ such that

$$x \in \mathcal{C}(\Delta) \tag{15}$$

$$\lambda_S(v(S) - \Delta - x(S)) = 0 \tag{16}$$

$$\frac{\partial L(x,\lambda)}{\partial x_i} = 0 \text{ for all } i \tag{17}$$

Condition (17) can be rewritten as:

$$(\lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S) x_i = a \tag{18}$$

$$\text{with } a = \prod_{i \in N} x_i, \tag{19}$$

Proof of Lemma: If we can find $a > 0, \lambda, x$ that satisfy (15),(16) and (18), then multiplying λ by a constant, we can also ensure that we find λ, x that satisfy (19) as well, hence x must be the solution to (P_Δ). Finally, we have:

$$L(x,\lambda,\Delta') = L(x,\lambda,\Delta) + (\lambda_N - \sum_S \lambda_S)(\Delta - \Delta').$$

Since $\mathcal{N}(\Delta) = \min_\lambda \max_x L(x, \lambda, \Delta)$, we get that if $\gamma = \lambda_N - \sum_S \lambda_S > 0$, then for all $\Delta' < \Delta$, $\mathcal{N}(\Delta') \geq \mathcal{N}(\Delta) + \gamma(\Delta - \Delta')$; and if $\gamma \leq (<)0$, then for all $\Delta' \geq \Delta$, $\mathcal{N}(\Delta') \geq (>)\mathcal{N}(\Delta)$.

Proof of Proposition 3: Consider the program (P_0) and x_0^* its unique solution. Condition (P1) implies that $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S \geq 0$, because otherwise, there would exists $\Delta_0 > 0$ such that $\mathcal{N}(\Delta) > \mathcal{N}(0)$ for all $\Delta \in (0, \Delta_0)$, hence, since $\mathcal{N}(\Delta)$ is continuous, $(P1)$ could not hold.

We distinguish two cases. If $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S > 0$, then we set $\eta = \lambda_N - \sum_{S \in \mathcal{S}} \lambda_S$ and $\mu_S = \lambda_S / \eta$. Then, since

$$\begin{aligned} \lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S &= \lambda_N - \sum_{S \in \mathcal{S}} \lambda_S + \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S \\ &= \eta(1 + \sum_{S \in \widetilde{\mathcal{S}}_i} \mu_S) \end{aligned}$$

equations (18) imply the desired equality:

$$(\alpha + m_i^\mu) x_i = (\alpha + m_1^\mu) x_1 \text{ for all } i. \tag{20}$$

with $\alpha = 1$.

If $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S = 0$, then we define $\mu_S = \lambda_S$ and obtain:

$$\lambda_N - \sum_{S \in \mathcal{S}_i} \lambda_S = \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S = \sum_{S \in \widetilde{\mathcal{S}}_i} \mu_S$$

hence equations (18) now imply the desired equality (20) with $\alpha = 0$. **Q. E. D.**

Proof of Claim A': Consider the program (P_0) and x^* its unique solution. Adding all equalities (18) yields

$$na = \lambda_N \sum_{i \in N} x_i^* - \sum_i \sum_{S \in \mathcal{S}_i} \lambda_S x_i^* = \lambda_N x^*(N) - \sum_{S \in \mathcal{S}} \lambda_S x^*(S) = \lambda_N v(N) - \sum_{S \in \mathcal{S}} \lambda_S v(S),$$

where the last equality follows from (16). It follows that

$$x_i^* = \frac{1}{n} \frac{(\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S) v(N) + \sum_{S \in \mathcal{S}} \lambda_S (v(N) - v(S))}{(\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S) + \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S}$$

As explained earlier, condition (P1) implies that $\lambda_N - \sum_{S \in \mathcal{S}} \lambda_S \geq 0$. Since $v(N) - v(S) \geq \eta$, and since $\sum_{S \in \mathcal{S}} \lambda_S \geq \sum_{S \in \widetilde{\mathcal{S}}_i} \lambda_S$, we thus obtain $x_i^* \geq \frac{\eta}{n}$. It follows that if for all $x \in \mathcal{C}$, $x_i < \frac{\eta}{n}$ for some i, then (P1) cannot hold.. **Q. E. D.**

Proof of Claim A: Claim A is a corollary of Claim A'. Nevertheless, we provide here a direct constructive proof of Claim A.

Define

$$\eta = v_* - \max_{S \neq N} v(S).$$

Consider the Nash maximization program under the core constraint C^ε, where $C^\varepsilon = \{x, x(N) = v_* + \varepsilon \text{ and } x(S) \geq v(S) \text{ for all } S \neq N\}$. Call x^*_ε the optimal solution and S^*_ϵ the set of coalitions that bind.

We first show that for any $S \neq N$ that contains I_1, S cannot bind for ε small enough. By contradiction, assume S binds. Then $x^*_\varepsilon(S) = v(S) \leq v_* - \eta$. Now observe that for any $I' \subset I_0$,

$$\max_{x \in C^\varepsilon} x(I')$$

is continuous in ε and takes value 0 at $\varepsilon = 0$. Thus for ε small enough, $x(N/S) \leq \eta/2$ for all $x \in C^\varepsilon$, hence $x^*(N) \leq v_* - \eta + \eta/2 < v_*$. Contradiction.

We now show that $P1$ fails to hold at any such ε, fixed. For Δ small compared to ε, define C^ε_Δ as the core corresponding to the value function $v = (v_{-N} - \Delta, v_* - \Delta + \varepsilon)$, $x^*_{\Delta,\varepsilon}$ the corresponding optimal solution of the Nash maximization program. We need to show that for Δ small, $x^*_{\Delta,\varepsilon}$ yields a higher Nash product.

Step 1: there exists $\eta_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, $x^*_{\varepsilon,i} \geq \eta_0$ for all $i \in I_1$.

For each $i \in I_1$, there exists $x^{(i)} \in C^0$ such that $x_i > 0$. Since C^0 is convex, $x = \frac{1}{I_1} \sum x^{(i)}$ is also in C^0 and has $\min_{i \in I_1} x_i \geq \eta_0$ for some η_0. Conclusion follows from the fact that $\max_{x \in C^\varepsilon} \min_{i \in I_1} x_i$ is continuous in ε.

Step 2: Define $a = \frac{\Delta}{|I_1|-1}$ and $b = \frac{1}{N} a$. Consider the allocation x obtained by substracting a to each individual in I_0 and adding b to the others. We claim that $x \in C^\varepsilon_\Delta$ and that its Nash product exceeds that of x^*_ε.

$x \in C^\varepsilon_\Delta$. Since we choose Δ small, we may restrict attention to coalitions S that bind, that is $S \in S^*_\epsilon$. Since S binds, not all $i \in I_1$ belong to S. So

$$\begin{aligned} x(S) &\geq x^*_\varepsilon(S) - (\mid I_1 \mid -1)a \\ &\geq x^*_\varepsilon(S) - \Delta \end{aligned}$$

and

$$\begin{aligned} x(N) &\leq x^*_\varepsilon(S) - a \mid I_1 \mid + b \mid I_0 \mid \\ &\leq x^*_\varepsilon(S) - a \mid I_1 \mid + a \leq x^*_\varepsilon(S) \end{aligned}$$

So $x \in C^{\varepsilon}_{\Delta}$.

Finally, the Nash product of x exceeds that of x^*_ε for all $\varepsilon \leq \varepsilon_1$ for some $\varepsilon_1 \leq \varepsilon_0$, because, by step 1, for ε_1 sufficiently small, the marginal effect on the Nash product of the increase b is arbitrarily larger than the marginal effect of the reduction a. **Q. E. D.**

Proof of Claim B.

Call $S^* = \arg\max_{S \neq N} \frac{v(S)}{|S|}$. The coalitional Nash solution is

$$x_i = \begin{cases} \frac{v(S^*)}{|S^*|} \\ \frac{v(N)-v(S^*)}{n-|S^*|} \end{cases}$$

Clearly, condition P1 is satisfied given that $\Delta \rightarrow \left(\frac{v(S^*)-\Delta}{|S^*|}\right)^{|S^*|} \left(\frac{v(N)-v(S^*)}{n-|S^*|}\right)^{n-|S^*|}$ is decreasing in Δ. **Q. E. D.**

Proof of claim C.

Take the coalitional Nash bargaining solution x for some Δ_0 and consider $\Delta < \Delta_0$ (which corresponds to assuming that the surplus of all teams is increased by $\Delta_0 - \Delta$). The vector y defined as $y_{i^*} = x_{i^*} + \Delta_0 - \Delta$ and $y_i = x_i$ for all $i \neq i^*$ is an element of the Δ-core. Since $\prod_{j \in N} y_j > \prod_{j \in N} x_j$, we conclude that $\mathcal{N}(\Delta) > \mathcal{N}(\Delta_0)$, hence that P1 holds. **Q. E. D.**

Proof of claim D:

S**tep 1.** Consider a strictly convex game and a core allocation[44] x such that $x(S') = v(S')$ and $x(S'') = v(S'')$ for $S' \neq S''$ and S', S'' both different from N. Then either $S' \subset S''$ or $S'' \subset S'$.

Proof of step 1. Assume by contradiction that $S' \cap S'' \subsetneq S'$ and $S' \cap S'' \subsetneq S''$, Then,[45]

$$\begin{aligned} x^*(S' \cup S'') &= x^*(S') + x^*(S'') - x^*(S' \cap S'') \\ &= v(S') + v(S'') - x^*(S' \cap S'') \\ &\leq v(S') + v(S'') - v(S' \cap S'') \\ &< v(S' \cup S'') \end{aligned}$$

which contradicts the fact that $x^* \in \mathcal{C}(v)$. So either $S' \subset S''$ or $S'' \subset S'$. **Q. E. D.**

[44] It is well known that strictly convex games have a non-empty core.

[45] The first inequality follows from $x^*(S' \cap S'') \geq v(S' \cap S'')$ because x^* is a core allocation. The second inequality follows from strict convexity.

Step 2. There is a player who belongs to all credible coalitions. This follows from step 1.

Step 3. Property P1 is satisfied. This follows from Step 2 and the fact that when there is a key player, P1 must hold (see claim C).

Step 4. The set of credible coalitions is an increasing sequence $S^{(1)} \subset S^{(2)} ... \subset S^{(m)} \subset N$. This follows from Step 1.

Proof of Claim E. Property P1 holds because player 1 must belong to all credible coalitions. Let x^* be the coalitional Nash bargaining solution and consider $S \subsetneq N$. Assume by contradiction that $S \setminus \{i\}$ is credible. This implies that $x^*(S \setminus \{i\}) = v(S \setminus \{i\})$. Since $x^*(N) = v(N)$, and since $x^*(N) + x^*(S \setminus \{i\}) = x^*(N \setminus \{i\}) + x^*(S)$, we obtain,

$$x^*(N \setminus \{i\}) + x^*(S) = v(N) + v(S \setminus \{i\})$$

hence, using property (PBS):

$$x^*(N \setminus \{i\}) + x^*(S) < v(N \setminus \{i\}) + v(S)$$

The latter inequality implies that either $v(N \setminus \{i\}) > x^*(N \setminus \{i\})$ or $v(S) > x^*(S)$, contradicting the fact that x^* is a core allocation. **Q. E. D.**