Gradualism in Bargaining and Contribution Games*

Olivier Compte† and Philippe Jehiel‡

19 January 2001

Abstract

We model bargaining situations in which parties have the option to terminate the negotiation, resulting in a termination outcome that depends on the offers made in the negotiation phase. The key features of the model are that 1) making a concession in the negotiation phase increases the other party's termination option payoff and 2) the termination outcome induces an efficiency loss as compared with a negotiated agreement. The main finding is that the mere threat of termination forces equilibrium concessions in the negotiation phase to be gradual, and the degree of gradualism is characterized. The model also applies to contribution games in which partial projects can be implemented. Our findings are contrasted with those appearing in the literature.

Journal of Economic Literature Classification Numbers: C72, C78, D60.

*We thank David Frankel, George Mailath, Steve Matthews, Ariel Rubinstein, Jozsef Sakovics, Jean Tirole, Jörgen Weibull, Yoram Weiss, and seminar participants at Tel Aviv University, Séminaire Fourgeaud (Paris), CEPR Summer meeting at Gerzensee, 7^{th} World Congress of the Econometric Society Tokyo 1995, MIT-Harvard, Princeton, Mannheim, LSE, SED conference 1999, for helpful comments.

†C.E.R.A.S.-E.N.P.C., C.N.R.S. (URA 2036), 28 Rue des Saints-Pères, 75007 Paris, France, compte@enpc.fr.

‡C.E.R.A.S., Paris, (C.N.R.S. URA 2036), and U.C.L, U.K., jehiel@enpc.fr.

1 Introduction

The work of Rubinstein (1982) has clearly been very influential in bargaining theory. One fundamental input of Rubinstein (1982) is to add an explicit dynamic structure to the bargaining game (see also Stahl 1972). Instead of making once for all proposals, parties are viewed as being engaged in a dynamic process of offers and counter-offers which can either be accepted or rejected. Incorporating a dynamic structure is not merely an embellishment of the bargaining theory. It permits to deal with new questions that cannot be dealt with in a static framework such as the one proposed by Nash (1950). A question that has attracted theorists is whether or not the bargaining strategies followed in equilibrium induce delays before an agreement can be reached, and if yes what is the magnitude of this delay. An even simpler issue is about understanding the dynamics resulting from the bargaining strategies: do the equilibrium strategies induce some form of gradualism? Do parties reach an agreement by going step by step? Under what circumstances should we expect a party to wait for the other party to make a further step before she does one more? These questions are - we believe - important toward understanding the relationship between theoretical approaches and real world bargaining in which such dynamics seem to prevail widely.

Rubinstein (1982) showed that the bargaining game with perfect information in which parties alternate in making offers to the other side, and in which each party discounts future payoffs according to some possibly party-specific discount rate has a unique Subgame Perfect Nash Equilibrium. Such a result was perceived as an instance in which game theory has predictive power. However, the unique equilibrium of this game does not induce a rich dynamics in terms of the employed strategies: in equilibrium there is immediate agreement (there is no delay), and strategies are stationary thus leaving no room for gradualism or other dynamic strategic effects.

Since then the literature on bargaining has moved on to see the effects of outside options (see Binmore et al. 1988), of incomplete information (see the survey of

Kennan and Wilson 1993, see also Abreu-Gul 2000, for recent work on this) and of both (see Compte-Jehiel 2000a).

Roughly speaking, the introduction of poor (stationary) outside options is shown to have no effect on the equilibrium strategies in a complete information setting, thus suggesting that stationary outside options alone cannot induce rich equilibrium dynamics.

In some of the approaches with incomplete information, there are sequential equilibria in which agreement is not reached immediately. However, most of these approaches heavily rely on the interpretations made off the equilibrium path about the private information held by the parties (there are signaling effects). A rich dynamics arises for some interpretations, but not for others, thus making it hard to derive sharp conclusions. Abreu and Gul (2000) (see also Compte-Jehiel 2000a) avoid the signaling effects by adopting a setup in which parties may with some positive probability behave mechanically according to some pre-specified obstinate pattern (i.e. always making the same demand). In sufficiently symmetric setups, this may result in an interesting dynamics in which (for a while) parties when rational mimic the behavior of an obstinate type with positive probability in the hope that the other party will reveal herself as rational first (thus resembling a war of attrition type of interaction). While interesting this kind of dynamics does not fully capture the common sense notion of gradualism in bargaining: in particular, it does not explain the commonly observed *pattern* of concessions that arise in real contexts.[1]

In this paper, we consider a complete information bargaining setting. The games we consider have two notable features: first, at any point in time, each party has the option to terminate the game; second and most importantly, in case of termination, the payoffs obtained by players depend on the history of offers or concessions made in the bargaining process. We show for a wide class of such situations that gradualism is a necessary feature of any equilibrium in which offers (or concessions) are made.

[1] In these models, equilibrium offers do not vary accross periods (only probabilities of acceptance do).

More precisely, we derive an upper bound on concessions that is shown to apply to any equilibrium of the game. And we show that this bound depends two parameters:

(i) The extent to which a concession made by a party increases his opponent's termination option payoff.

(ii) The inefficiency of the termination option.

The intuition is that the termination option is a threat that players have to take into account when making a concession: the termination option value acts as a lower bound on equilibrium payoffs, hence a player should be concerned about not raising too much the termination option value of his opponent. When concessions translate into an increased termination option value for the opponent, a player should be concerned about not making too large concessions.

Of course, if the termination option is very inefficient, then the threat of termination will not be strong: raising an opponent's termination option should not be a concern whenever terminating is a poor option anyway. But, for not too inefficient termination options, the threat of terminaing is active, and it structures the pattern of concessions, as we show.

We will shortly give two examples that will illustrate further the strategic reason leading to gradualism. Before that, let us emphasize that our interest in history-dependent options is not solely theoretical. We believe that there are many bargaining situations that fit in well with our assumptions.

Such situations include negotiations with conventional or final-offer arbitration because arbitrators are likely to take into account the offers made in the bargaining process to improve their decisions.

They also include bargaining contexts in which it is possible to postpone the negotiation on unsettled issues till a second round of negotiation (most trade negotiations between Europe and the US say have this feature) because then what has been agreed upon during the initial bargaining round is taken for granted in the second round.

Our model also fits well with bargaining situations where parties have to share the cost of a joint project and where financing takes the form of a contribution or subscription game. It corresponds to a situation where (i) parties alternate in making contributions up to some targeted level, say K ; and (ii) each player has the option to terminate the contribution process, in which case the total contribution collected, say $k < K$, is used to implement a smaller scale project.

Our finding about gradualism might at first glance seem reminiscent of the insight of Admati and Perry (1991). They consider a game in which agents alternate in making contributions, assumed to be sunk. When the total contribution reaches the amount required to finance the public project, the project is implemented and agents enjoy benefits accordingly. They show in a complete information setting assumed to be symmetric (costs and benefits are assumed to be the same for all agents) that in equilibrium agents make their contributions step by step (when costs are weakly convex).

Despite the common feature of gradualism, the logics of their finding is rather different from ours - there is no analog to the threat of termination in their model. Furthermore, as we show elsewhere (Compte and Jehiel 2000b) their insight about gradualism is very sensitive to the symmetry assumption whereas symmetry plays no role in our framework. (See the discussion section.)

To conclude this introduction, we wish to give two simple examples that illustrate how the threat of termination leads to gradualism, and that also illustrate why our insight applies both to bargaining and contribution games.

(i) a simple alternating offer game:

Two parties alternate in making offers about the splitting of a pie of size one. They discount future payoffs with the same discount factor (close to 1, say). In every other period, a party may either make an offer to the other party or he may decide to opt out. Opting out is assumed to cost him γ and to result in a compromise partition that lies in between the most generous offers made so far by the two parties, say the one that is at equal distance from these two offers.

Our main result will show that, in any equilibrium, parties never increase their most generous offer by more than 4γ (in a single round): parties typically start by making offers that are very favorable to them and then gradually reduce their demand step by step, by an amount that cannot exceed 4γ.

The (strategic) reason for gradualism here is the fear that the other party might take advantage of a compromise partition that would be offered too quickly. For the sake of illustration, suppose in the game just described that party 1 proposes right away a $(1/2, 1/2)$ split to party 2 (as in Rubinstein's model). For γ small enough (i.e. smaller than $1/8$), party 2 will not accept that offer. Indeed, so far party 2 has not made any offer. So by opting out, party 2 can guarantee[2] herself $3/4 - \gamma$ which is strictly more than what party 2 would obtain by accepting the $(1/2, 1/2)$ proposal of party 1. Thus in equilibrium party 1 cannot expect party 2 to accept the $(1/2, 1/2)$ offer.

As a matter of fact, in equilibrium, it cannot be optimal for party 1 to offer $(1/2, 1/2)$ right away (for γ smaller than $1/8$) because such a move allows party 2 to secure $3/4 - \gamma$ as we have just seen, and this minimal payoff that party 2 can guarantee herself also implies that party 1 will never be able to obtain more than $1 - (3/4 - \gamma)$ in such a scenario. But by opting out right away, party 1 can get $1/2 - \gamma$. When γ is smaller than $1/8$, $1/2 - \gamma$ is strictly greater than $1/4 + \gamma$, hence opting out right away is a better option for party 1 than offering $(1/2, 1/2)$. The same (two-step) argument can actually be used to show that a party's most generous offer cannot increase by more than 4γ in a single round: the pattern of offers and counter-offers must be gradual in equilibrium.

(ii) a simple contribution game.

Two players alternate in making contributions to a joint project. The targeted total contribution is equal to K, in which case the joint project of size K is implemented, yielding each player a gain equal to $aK/2$. In every other period, a party

[2]Note that $3/4 = \frac{1}{2}(1/2+1)$ stands for unweighted average between $1/2$ (the most generous offer of 1) and 1 (the demand of 2).

may either make a contribution or decide to terminate the contribution process. In the latter case, the total contribution made, say $k < K$, is used to implement a smaller scale project, yielding each player a gain equal to $\frac{b}{2}k$ where $b < a$. The cost of contributing c is assumed to be equal to c, and we assume that $a > 1$. So the large scale project generates a surplus equal to $aK/2 + aK/2 - K = (a-1)K$.

Suppose that player 1 starts by contributing $K/2$ right away (expecting player 2 to do the same). Then by terminating, player 2 may secure a payoff equal to $bK/4$. By contributing $K/2$, she would get a payoff equal to $(a-1)K/2$. So whenever $(a-1) < b/2$ she will prefer the termination option, and party 1 cannot expect party 2 to reciprocate and contribute $K/2$.

As a matter of fact, if $a-1 < b/4$, it cannot be optimal for player 1 to contribute $K/2$ right away. Since player 2 may secure a payoff equal to $bK/4$ by terminating, and since the maximum surplus that can be derived is equal to $(a-1)K$, the maximum payoff that player 1 may obtain when he contributes $K/2$ is $(a-1)K - bK/4$, which is smaller than 0 when $a-1 < b/4$. Thus making no contribution throughout the game (or terminating the contribution process immediately) is a better option for party 1.

The rest of the paper is organized as follows. In Section 2 we present our model. The main results are presented in the first part of Section 3. Subsection 3.2 suggests how a number of applications fit in well in our setup. Section 4 includes a discussion; in particular, it reviews the insights derived in the contribution game literature and shows how these differ from (relate to) our finding.

2 The Model

In this Section, we present a model that may either be interpreted as a bargaining game or as a contribution game. In the first case, parties make concessions, while in the latter case, they make contributions. To clarify exposition, we choose to present the model as a contribution game, and next (see examples) give the bargaining interpretation of the game.

There are two players $i = 1, 2$. Players take turns in making voluntary contributions, starting with player 1 in period 1. Formally, we shall denote by c^t the contribution made by the player who may contribute at t, and by c_i^t the contribution made by player i at t.[3] We shall also denote by x_i^t the total contribution made by player i up to and including date $t-1$, by $x^t = (x_1^t, x_2^t)$ the profile of total contributions, and by k^t the aggregate total contribution made by both players:

$$x_i^t = \sum_{s<t} c_i^s. \tag{1}$$

$$k^t = x_1^t + x_2^t \tag{2}$$

We shall assume that the aggregate total contribution cannot exceed K, hence at any date where a player, say i, moves, his contribution cannot exceed $K - k^t$. This assumption is not critical. It is made to facilitate the bargaining interpretation of the model.

The key feature of our model is that each player has the option to terminate the contribution game. Formally, at any date t where player i moves, he may either make a contribution c_i^t (possibly equal to 0, and no larger than $K - k^t$) or he may terminate the contribution game. When he chooses to contribute, the game proceeds to date $t + 1$, where it is player j's turn to move. When he chooses to terminate the game, we adopt the convention that $c_i^t = 0$. There are thus two instances under which the game ends: it occurs either when the aggregate total contribution reaches K, or when a player voluntarily terminates the game.

Consider a date t. A $t-$history, denoted h^t, is a sequence of contributions $(c^1, ..., c^{t-1})$ made by the agents prior to time t. The total contribution made along history h^t is denoted $k(h^t) \equiv \sum_{s<t} c^s$. The set of $t-$histories is denoted H^t. It will also be convenient to let H_0^t denote the set of $t-$histories for which the total contribution

[3] We adopt the convention that when no contribution is made at t (respectively when player i does not contribute at t), then $c^t = 0$ (respectively $c_i^t = 0$).

Thus, if party i moves at t, then $c_i^t = c^t$ and $c_j^t = 0$.

has not reached K yet. We have:

$$H^t \equiv \{h^t = (c^1, ..., c^{t-1}), k(h^t) \leq K\}, \text{ and}$$
$$H_0^t \equiv \{h^t = (c^1, ..., c^{t-1}), k(h^t) < K\}$$

A strategy for player i specifies, for each date t at which it is player i's turn to move and after any $t-$history $h^t \in H_0^t$, whether to terminate the game or to contribute, and the size of the contribution c_i^t in the latter case.

An outcome of the game, which we denote ω, is a (possibly infinite) termination date, denoted T, and a profile of sequences of contributions (of length T):

$$\omega \equiv (T, \{c_1^s\}_{s=1}^T, \{c_2^s\}_{s=1}^T)$$

The set of outcomes is denoted Ω, and the set of outcomes for which termination occurs at T is denoted Ω^T.[4] For any given outcome $\omega \in \Omega$, it will be convenient to denote by x_i^ω (resp. $k^\omega = x_1^\omega + x_2^\omega$) the total contribution made by i (resp. the total aggregate contribution made by players) under ω.

Also, for any $h^t \in H^t$, we denote by $\omega^{term}(h^t)$ the outcome induced by h^t, under the assumption that termination is triggered at t in case $k(h^t) < K$.[5]

Finally, for $i = 1, 2$, and for any outcome $\omega \in \Omega$, player i's payoff is given by $u_i(\omega)$. And for any T, $\omega \in \Omega^T$, and for any $t \leq T$, we denote by $u_i^t(\omega)$ the continuation payoff derived from ω by player i, *from date t on*.

In the rest of this Section, we provide four examples (two contribution games and two bargaining games) that are special cases of the general model presented above, and for which the payoff fonctions $u_i(\omega)$ and $u_i^t(\omega)$ are specified.

[4] Note that by contruction, an outcome $\omega \in \Omega^T$ satisfies one of the two following properties:

(i) either $k^\omega < K$, in which case termination is trigerred at T, and we have $c_1^T = c_2^T = 0$.

(ii) Or $k^\omega = K$, in which case the last player to contribute, say i, chose $c_i^T = K - k^T > 0$.

[5] Note that if $k(h^t) = K$, the project has been completed before t. The termination date associated with $\omega^{term}(h^t)$ corresponds to the last date (before t) where a positive contribution was made.

Contribution game 1: Consider the case where the project implemented only depends on the total contribution of both players. Let $v_i(k)$ be the valuation to player i when the total contribution is k. Assume that contributions are sunk and that the cost incurred by player i when contributing c_i is exactly c_i.[6] Also assume that both players are impatient, and discount both contributions and benefits using a discount factor $\delta < 1$. For any outcome $\omega = (T, (\{c_1^s\}_{s=1}^T, \{c_2^s\}_{s=1}^T))$, we have

$$u_i(\omega) = \delta^{T-1} v_i(k^\omega) - \sum_{1 \leq s \leq T} \delta^{s-1} c_i^s.$$

and

$$u_i^t(\omega) = \delta^{T-t-1} v_i(k^\omega) - \sum_{t \leq s \leq T} \delta^{s-t-1} c_i^t.$$

Contribution game 2: Consider the same setup as in contribution game 1, except for the sunk cost assumption: assume now that c_i is a pledge to contribute made by player i. Then for any outcome $\omega \in \Omega^T$, we have:

$$u_i(\omega) = \delta^{T-1}[v_i(k^\omega) - x_i^\omega].$$

and

$$u_i^t(\omega) = \delta^{T-t-1}[v_i(k^\omega) - x_i^\omega].$$

Bargaining game 1: Two parties bargain on a pie of size $K = 1$. Any chunk of size c of the pie is valued c by each party. The contribution c_i^t is now interpreted as a concession made by party i at t, and x_i^t is interpreted as the total concession made by party i to party j up to (and not including) date t. At any date where he moves, party i may either make a concession or opt out. The game ends either when one party opts out or when the total concession $k^t = x_1^t + x_2^t$ reaches 1. The value of the outside option to party i is assumed to depend on the profile of total contributions $x^t = (x_1^t, x_2^t)$ made by the parties, and we denote it

$$v_i^{out}(x^t).$$

[6] Admati-Perry (1991) considers convex costs, see the discussion section.

Also assume that both players are impatient, and discount both contributions and benefits using a discount factor $\delta < 1$. Finally assume that each concession c_i is a commitment to give a chunk c_i of the pie in case an agreement is reached.

Preferences of the parties over outcomes are described by:

$$u_i(\omega) = \delta^{T-1}[v_i(x^\omega)].$$

and

$$u_i^t(\omega) = \delta^{T-t-1}[v_i(x^\omega)].$$

where v_i is a function defined over profiles of total contributions, such that:

$$\begin{aligned} v_i(x) &= 1 - x_i \text{ if } x_1 + x_2 = 1, \text{ and} \\ v_i(x) &= v_i^{out}(x) \text{ otherwise} \end{aligned}$$

Bargaining game 2: Consider the same setup as in bargaining game 1, but assume now that c_i can be consumed right away by party i. Also assume that party i derives an extra-payoff equal to v_i^0 in case the negotiation terminates and the outside option is not triggered. The preferences of the parties over outcomes are now represented by:

$$u_i(\omega) = \sum_{1 \le s \le T} \delta^{s-1} c_j^s + \delta^{T-1} v_i(x^\omega)$$

and

$$u_i^t(\omega) = \sum_{t \le s \le T} \delta^{s-1} c_j^s + \delta^{T-1} v_i(x^\omega)$$

where $v_i(.)$ is now defined by:

$$\begin{aligned} v_i(x) &= v_i^0 \text{ if } x_1 + x_2 = 1, \text{ and} \\ v_i(x) &= v_i^{out}(x) \text{ otherwise} \end{aligned}$$

3 The main result.

The key aspect of our model is the following. We have defined a model of contribution in which throughout the process, the players have the option of terminating the contribution process, and implementing an outcome that depends on the total contributions made so far.

This termination option is a threat that players have to take into account when making a contribution, and this Section will make clear how this threat affects the contribution that players make in equilibrium. Before presenting our main result, we make two observations that will give some preliminary intuition for our result.

A first observation is that a large contribution may induce a large increase in the value that one's opponent may derive from terminating. And intuitively, a player should be concerned about not raising too much this option value for the opponent: the option value acts as a lower bound on equilibrium payoffs.

A second observation is that if the termination option is very inefficient, then presumably the threat of termination will not be strong: raising an opponent's termination option should not be a concern in a case where terminating is a poor option anyway.

Our main result will confirm that in determining the size of his contribution, a player should be concerned with (i) the extent to which he increases his opponent's option value when he contributes, and (ii) the inefficiency of the termination option. It will also quantify these two effects.

To present our result formally, it will be convenient to define the following.

Consider any date t where, say, player i moves, and any history of contribution $h^t \in H^t$. For any $\tau \geq 0$, we denote by $(h^t, c^t, ..., c^{t+\tau}) \in H^{t+\tau+1}$ the sequence of contributions that starts with h^t and continues with c^t at t, c^{t+1} at $t+1....$, $c^{t+\tau}$ at $c^{t+\tau}$. We denote by $H(h^t)$ the set of sequences of contributions that start with h^t.

For any date T and outcome $\omega \in \Omega^T$, and for any date $t \leq T$, we let $W^t(\omega)$ denote the welfare (as viewed from date t) associated with ω:

$$W^t(\omega) = u_i^t(\omega) + u_j^t(\omega) \tag{3}$$

We also let $\bar{W}^t(h^t, c)$ denote the maximum welfare (as viewed from date t) that players may derive in case the history of contribution is h^t and player i makes a contribution c at t. That is, we define:

$$\bar{W}^t(h^t, c) = \sup_{h \in H(h^t, c)} W^t(\omega^{term}(h))$$

We are now ready to state our main result:

Proposition 1 *Consider any Subgame perfect equilibrium of the game described in Section 2. At any date t where it is player i's turn to move, after any history h^t, if player i makes a contribution equal to c, then, we must have:*

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t)) \leq \bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t))$$

The difference $u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t))$ represents the change in party j's (date t) continuation payoff that is induced when player i contributes c and player j triggers termination at date $t+1$ as compared with player i terminating at t. So our proposition says that, in equilibrium, the contribution c may increase the value of the termination option for player j, but only to the extent that the increase does not exceed the efficiency loss that would result from termination. This in turn allows us to derive an upper bound on the equilibrium contributions made by the players. The bounds will be made specific in the next subsections.

The argument of the proof is to compare what player i obtains when he terminates with what he may obtain at most if he makes a contribution c. If player i makes a large contribution c, it does not satisfy the above constraint. Player j's termination option becomes very valuable to player j. And possibly so valuable that player i may only end up with a small payoff in equilibrium, thereby making the option to terminate more attractive (relative to making the large contribution).

Proof. Consider a Subgame perfect equilibrium, a date t where it is player i's turn to move, and a history of contributions h^t. Assume that in equilibrium, player i makes a contribution equal to c with positive probability, and let (v_1, v_2) denote players' continuation equilibrium payoffs associated with that contribution.

Since player i has the option to terminate at t, we must have

$$v_i \geq u_i^t(\omega^{term}(h^t)). \quad (4)$$

Since player j has the option to terminate at $t+1$ (after player i's contribution), we must have:

$$v_j \geq u_j^t(\omega^{term}(h^t, c)). \tag{5}$$

Since, when player i contributes c, player i and j cannot expect to get a joint payoff higher than $\bar{W}^t(h^t, c)$ (by definition of $\bar{W}^t$), we have:

$$v_i + v_j \leq \bar{W}^t(h^t, c) \tag{6}$$

The Proposition follows from combining the above inequalities and from the definition W^t. ■

Before proceeding to the applications, we give a slightly different version of Proposition 1 that will simplify exposition in the next Section, and that will enable to find bounds on contributions that are independent of the particular specification of time preferences.

To this end, we make two assumptions that will always be met in applications, but that the general structure so far does not imply. First, we assume that a player is better off terminating after his opponent makes a positive contribution rather than a nul contribution:

Assumption 1: *For any $i, j \in \{1, 2\}$, $j \neq i$, for any t where player i moves, any history of contributions h^t, any contribution $c \in [0, K - k^t]$, and any $t' \leq t$,*

$$u_j^{t'}(\omega^{term}(h^t, c)) - u_j^{t'}(\omega^{term}(h^t, 0)) \geq 0$$

In most cases, if termination results in a positive payoff, then a player prefers that termination occurs now rather than one period later. For those instances in which this is not the case, our second assumption stipulates that if a player prefers termination to be delayed one period, then he prefers an even longer delay, which again is met in applications:

Assumption 2: *For any t* and *$h^t \in H^t$, and any $t' \leq t$,*

$$u_i^{t'}(\omega^{term}(h^t)) \leq u_i^{t'}(\omega^{term}(h^t, 0)) \Longrightarrow u_i^{t'}(\omega^{term}(h^t, 0)) \leq u_i^{t'}(\omega^{term}(h^t, 0, 0)).$$

Under Assumptions 1 and 2, we have:

Proposition 2 *Consider any Subgame perfect equilibrium. At any date t where it is player i's turn to move, after any history h^t, if player i makes a partial contribution equal to c, then, we must have:*

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t, 0)) \leq \bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t, 0))$$

Proof. The proof of this Proposition is identical to that of Proposition 1, except for inequality (4). We will now show that

$$v_i \geq u_i^t(\omega^{term}(h^t, 0)), \tag{7}$$

which, along with inequalities (5) and (6), will permit us to conclude. We distinguish two cases. Either $u_i^t(\omega^{term}(h^t)) \geq u_i^t(\omega^{term}(h^t, 0)$, in which case (4) implies (7). Or $u_i^t(\omega^{term}(h^t)) \leq u_i^t(\omega^{term}(h^t, 0))$, in which case, by Assumption 1 and 2:

$$u_i^t(\omega^{term}(h^t)) \leq u_i^t(\omega^{term}(h^t, 0)) \leq u_i^t(\omega^{term}(h^t, 0, 0)) \leq u_i^t(\omega^{term}(h^t, 0, c'))$$

for all $c' \geq 0$. Inequality (7) holds because player i has the option to make no contribution at t and terminate at $t+2$, so either player j terminates at $t+1$ and then player i obtains $u_i^t(\omega^{term}(h^t, 0))$, or player j contributes $c' \in [0, K - k^{t+1})$, and then player i gets $u_i^t(\omega^{term}(h^t, 0, c')) \geq u_i^t(\omega^{term}(h^t, 0))$, or player j contributes $c' = K - k^{t+1}$, and then player i gets $u_i^t(\omega^{term}(h^t, c')) \geq u_i^t(\omega^{term}(h^t, 0))$ (by Assumption 1). ■

We now turn to the implication that Propositions 1 and 2 have in the bargaining and contribution applications.

3.1 Contribution games.

We consider contribution games 1 and 2 simultaneously.

For any total aggregate contribution k, we define

$$\gamma(k) \equiv \sup_y v_1(k+y) + v_2(k+y) - y - v_1(k) - v_2(k)$$

That is, $\gamma(k)$ corresponds to (an upper bound on) the efficiency loss associated with termination. We also let

$$\lambda_i = \inf_{k, 0<y\leq K-k} \frac{v_i(k+y) - v_i(k)}{y}.$$

The interpretation of λ_i is that any increase of y of the total contribution increases player i's termination option payoff by $\lambda_i y$ at least. In applications, λ_i is likely to be strictly positive as both players are likely to benefit from the implementation of a bigger project. In what follows, we assume that $\lambda_i > 0$ for each player i.

Proposition 3 *Consider any subgame perfect equilibrium of the game. Let t be a date where player i moves and let k^t be the corresponding total contribution made up to t. Player i never contributes more than $\frac{\gamma(k^t)}{\lambda_j}$.*

Proof. We consider first the case where player i makes a contribution c equal to $K - k^t$, and apply Proposition 1. We have

$$\bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t)) \leq \gamma(k^t)$$

and

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t)) = v_j(k^t + c_i^t) - v_j(k^t) \tag{8}$$

In case player i makes a partial contribution, e.g. $c < K - k^t$ we apply Proposition 2. We now have

$$\bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t, 0)) \leq \delta\gamma(k^t)$$

and

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t, 0)) = \delta[v_j(k^t + c) - v_j(k^t)]$$

So in both cases,

$$v_j(k^t + c) - v_j(k^t) \leq \gamma(k^t)$$

which implies the desired bound on c since $v_j(k^t + c) - v_j(k^t) \geq \lambda_j c_i^t$ by definition of λ_j. ■

3.2 Bargaining games.

Game 1. For any total concession profile x, we define

$$\gamma(x) = 1 - v_1^{out}(x) - v_2^{out}(x)$$

That is, $\gamma(x)$ corresponds to the efficiency loss associated with the outside option when the current total concession profile is x: players would share a surplus equal to 1 in case of agreement, whereas they get $v_1^{out}(x) + v_2^{out}(x)$ by opting out. We also let

$$\lambda_i(x) = \inf_{k, 0<y<1-x_1-x_2} \frac{v_i^{out}(x_i, x_j + y) - v_i^{out}(x_i, x_j)}{y}.$$

We assume that $\lambda_i > 0$.

Finally, we assume that for each $i, j \neq i$,

$$v_i^{out}(x_i, x_j) \leq 1 - x_i \text{ for all } x_j < 1 - x_i, \tag{9}$$

This assumption is not key to the gradual concession result that follows. It permits to get a simple expression for the bound on the last concession made. It says that opting out is less attractive than having his opponent concede the rest, even when there remains very little to concede. One interpretation is that in the outside option the concessions made are confirmed, so player i cannot hope to get a share of the pie larger than $1 - x_i$ (in addition there may be some extra costs to opting out).

Proposition 4 *Consider any subgame perfect equilibrium of the game. At any date t where he moves, player i never contributes more than $\frac{\gamma(x^t)}{\lambda_j(x^t)}$.*

Before proceeding to the proof, we provide a simple example where the assumptions above are satisfied.

Example 1 *Assume that when termination is triggered, (i) past concessions are confirmed, (ii) what has not been conceded is shared equally between parties, and (iii) each party bears a cost γ_0. That is,*

$$v_i^{out}(x) = x_i + \frac{1 - x_1 - x_2}{2} - \gamma^0.$$

Then we have:

$$\lambda_i(x) = \frac{1}{2} \text{ and } \gamma(x) = 2\gamma^0$$

And by Proposition 4, concessions can never exceed $4\gamma^0$ *at any point in time.*

Proof of Proposition 4: Upper bounds on $\bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t))$ and $\bar{W}^t(h^t, c) - W^t(\omega^{term}(h^t, 0))$ are respectively equal to $\gamma(x^t)$ and $\delta\gamma(x^t)$. For $c < 1 - k^t$, we have

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t, 0)) = \delta[v_j^{out}(x_j^t, x_i^t + c) - v_j^{out}(x_j^t, x_i^t)]$$

which implies the desired result.

Besides, when $c = 1 - k^t$, we have

$$u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t)) = 1 - x_j^t - v_j^{out}(x_i^t, x_j^t)$$

and for any $y < c = 1 - k^t$,

$$1 - x_j^t - v_j^{out}(x_i^t, x_j^t) \geq v_j^{out}(x_j^t, x_i^t + y) - v_j^{out}(x_j^t, x_i^t) \geq \lambda_j y$$

Since y can be chosen arbitrarily close to c, we obtain the desired bound on c. ■

Game 2: For any total concession profile x, we define

$$\gamma(x) = 1 - x_1 - x_2 + v_1^0 + v_2^0 - v_1^{out}(x) - v_2^{out}(x)$$

That is, at x, the share $x_1 + x_2$ has been consumed, and there only remains $1 - x_1 - x_2$ to consume. In the event an agreement is reached, players would therefore share a surplus equal to $1 - x_1 - x_2 + v_1^0 + v_2^0$. Thus $\gamma(x)$ corresponds to the efficiency loss associated with the outside option when the current total concession profile is x.

We define $\lambda_j(x)$ as follows:

$$\lambda_i(x) = \inf_{k, 0 < y < 1 - x_1 - x_2} \frac{y + v_i^{out}(x_i, x_j + y) - v_i^{out}(x_i, x_j)}{y}$$

and we assume that

$$v_i^{out}(x) \leq 1 - x_1 - x_2. \tag{10}$$

That is, a player cannot hope to get more than what remains to be consumed by opting out.

We have the following Proposition:

Proposition 5 *Consider any subgame perfect equilibrium of the game. At any date t where he moves, player i never concedes more than $\frac{\gamma(x^t)}{\lambda_j(x^t)}$.*

Once again, before proceeding to the proof, we provide a simple example where the assumptions above are satisfied.

Example 2 *Assume that when termination is triggered, what has not been conceded is shared equally between parties, and that each party bears a cost γ_0. That is,*

$$v_i^{out}(x) = \frac{1 - x_1 - x_2}{2} - \gamma^0.$$

Then we have:

$$\lambda_i(x) = \frac{1}{2} \text{ and } \gamma(x^t) = 2\gamma^0 + v_1^0 + v_2^0$$

Proposition 5 thus implies that in any subgame perfect equilibrium, concessions can never exceed $4\gamma^0 + 2(v_1^0 + v_2^0)$.

Proof. The proof is identical to the previous one, except for the case where player i concedes $c = 1 - k^t$. Then, since $v_j^{out}(x_i^t + c - \varepsilon, x_j^t) \leq \varepsilon$ by (10):

$$\begin{aligned} u_j^t(\omega^{term}(h^t, c)) - u_j^t(\omega^{term}(h^t)) &= c + v_j^0 - v_j^{out}(x_i^t, x_j^t) \\ &\geq c - \varepsilon + v_j^{out}(x_i^t + c - \varepsilon, x_j^t) + v_j^0 - v_j^{out}(x_i^t, x_j^t) \\ &\geq \lambda_i(x^t)(c - \varepsilon) + v_j^0 \geq \lambda_i(x^t)(c - \varepsilon). \end{aligned}$$

Since ε can be chosen arbitrarily small, we may conclude. ■

Propositions 4 and 5 derive upper bounds on equilibrium concessions. However, they do not address the issue whether players do make concessions in equilibrium.

In Appendix we develop Example 2 above and exhibit an equilibrium for this specification in which players do make concessions and the bound derived in Proposition 5 is tight.

4 Discussion

4.1 Related literature on gradualism

Admati and Perry (1991) were the first to consider a dynamic contribution game leading to gradual contributions in equilibrium.[7] Their game corresponds to contribution game 1 except that 1) Agents have no access to a partial completion project and 2) The cost of contributing c is $\phi(c)$ where $\phi(\cdot)$ is assumed to be an arbitrary convex function. And they analyze the symmetric case in which both agents value the project in the same way, both agents are equally patient and incur the same costs when making the same contribution. Their main insight is that if the project is completed, several small steps are made in equilibrium. They also observe that if the contributions are not sunk but are conditional on the completion of the project, there are two large contributions in equilibrium, thus suggesting that the sunk character of contributions is the key determinant of gradualism in contribution games.

In a companion paper (Compte-Jehiel 2000b), we revisit Admati-Perry's contribution game and relax the assumption that agents are symmetric. To fix ideas, we assume that agent 1's valuation of the project V_1 is higher than agent 2's valuation V_2. And we assume that agents are otherwise symmetric and that the cost of contributing c is c for both agents (in fact even a not too convex function $\phi(\cdot)$ would yield the same conclusion). As in Admati-Perry, we find that there is a unique subgame Perfect Nash equilibrium. However, in contrast with Admati-Perry, the equilibrium has now a very different form: Agent 2 is the first to make a (positive) concession

[7]Bliss and Nalebuff (1984) is an early work that also considers a dynamic contribution game, but in which partial contributions are not admitted: each player may either contribute nothing or pay for the entire public good. As a result, their game has the structure of a war of attrition.

equal to $K - V_1$. Then agent 2 contributes V_1. Thus breaking the symmetry between agents invalidates the conclusion that when contributions are sunk the equilibrium pattern is gradual.

Compte-Jehiel (2000b) also considers a model in which agents are symmetric, but the concessions are partially sunk in the sense that agents get a partial refund (proportional to their total contribution) when the project is not completed. Here again and whatever the magnitude of the refund, we observe that in equilibrium there are two big contributions. Thus, even in the symmetric case, a small amount of refund invalidates the conclusion that the equilibrium pattern is gradual.

The above insights should be contrasted with those derived in Section 3 when agents have access to a partial completion option. Then whether contributions are sunk or not (contribution games 1 and 2, respectively), and whether or not agents are symmetric (there is no symmetry assumption throughout the paper), equilibrium contributions must be small, thus showing that the partial completion option results in gradualism in contribution games.

Since Admati-Perry's paper, there have been a few papers dealing with contribution or bargaining games and irreversibilities. These include our earlier unpublished paper Compte-Jehiel (1995) and Marx-Matthews (2000), Lockwood-Thomas (1999), Gale (2000) and Compte-Jehiel (2000b).

Compte-Jehiel (1995) is a special case of the model developed in this paper that corresponds to bargaining game 1. The content of Compte-Jehiel (2000b) has already been discussed.

Marx -Matthews (2000) develops a contribution game and proposes a variant on Admati and Perry's contribution game, which differs from that in Compte-Jehiel (2000b). They consider a contribution game where players may make contributions *simultaneously* and players receive a flow of payoffs all along the contribution path that is a function of the current level of contribution. Their main focus is on efficiency. They show that the simultaneity in players' contributions may allow players to sustain efficient outcomes. The simultaneity plays a crucial role in their analysis in that it allows to sustain the no contribution outcome if a player deviates from a prescribed strategy. Note however that this threat fails to be credible in asymmetric settings.

Marx-Matthews' model does not fall in the class studied in this paper because in our setup agents move in alternate order and not simultaneously. However, consider the same setup as described in Section 2 in which agents move now simultaneously. If we restrict attention to equilibria in pure strategies, the same insight as the one in Proposition 1 applies. (From a given point in time, consider a perfect Nash equilibrium in which parties 1 and 2 make contributions (c_1, c_2) respectively. Agent j's continuation payoff is at least that obtained after not contributing and then terminating the game so that the same inequalities as in Proposition 1 can be obtained.) Thus, our insight of gradualism is robust to the timing of the game.[8]

Lockwood-Thomas (1999) consider a two-player dynamic game in which each player controls a one-dimensional variable interpreted as a level of cooperation. At any date, each player may invest in (i.e. increase) the level of cooperation. (This investment may be interpreted as a contribution). Again, they obtain a gradual pattern of investment in the level of cooperation, as not investing further plays much the same role as opting for termination in our model (this is analogous to our comment about Marx-Matthews - see footnote 8). Like Marx-Matthews, they consider a model in which players move simultaneously.

Gale (2000) considers the class of monotone games with positive spill-overs, in which again each player controls a one-dimensional variable (or state) that may only increase over time, and in which at each date, each player receives a payoff that

[8] In Marx-Matthews' model there is no partial completion option as in our contribution games 1 and 2. However, the flow of payoffs (that is a function of the total contribution made so far) plays a similar - though less transparent - role. The point is that at any point in time an agent has the option not to make any more contribution, which guarantees him a flow of payoff that is a function of the current level of contribution. This minimum payoff an agent can secure is similar to the termination option assumed in our setup.

depends on the current value of the state profile. Gale (2000) is concerned about characterizing the subgame perfect equilibrium paths of games in this class. As in the papers discussed above, not increasing further one's own state plays the same role as opting for termination in our model, and our gradual contribution result therefore applies to games in this class as well.[9]

4.2 On the bargaining interpretation of the termination option

In Introduction we have suggested two interpretations for the termination option in a bargaining context. We now elaborate a little bit on these.

The first interpretation is that of an arbitration clause. There is a huge literature in industrial relations about the various forms of arbitration. Following Stevens (1966), a key distinction is made between conventional arbitration and final offer arbitration. A typical example of conventional arbitration is one where the arbitrator systematically takes the average position between the (incompatible) positions held by the two parties. Note that to the extent that arbitrators' fees are independent of parties' positions, this fits exactly with Example 1 above.

Conventional arbitration has been criticized because of the *chilling effect* that it induces at the negotiation stage (see for example Feuille 1975). Viewing the negotiation stage as a one-shot interaction, the chilling effect expresses the idea that parties will tend not to make compromises in an attempt to improve their position at the arbitration stage. In some sense, our finding of gradualism is the dynamic

[9] To be more precise, let x_i denote the state of player i, and $u_i(x_i, x_{-i})$ the payoff received by player i at the current date if the current state profile is (x_i, x_{-i}). By not increasing the state any further, player i secures $u_i(x_i, x_{-i})$, which gives a lower bound on player i's equilibrium continuation payoff that plays the same role as $u_i^t(\omega^{term}(h^t))$ in our model.

Also note that in Gale's model, player i has an even better option. The worst scenario for player i is the one in which the other players do not make further contributions. And in that case, player i may still secure $\max_{x_i^0 \geq x_i} u_i(x_i', x_{-i})$ by contributing up to some x_i' and not contributing further in the rest of the game. This gives a lower bound on continuation payoff which is larger than $u_i(x_i, x_{-i})$, so, following our approach, a tigher bound on contributions could be obtained.

counterpart of the chilling effect identified in this literature.

Final offer arbitration has been introduced by Stevens as a remedy against the chilling effect. Now the arbitrator is no longer viewed as compromising between the positions held by the parties, but simply as picking one of the two positions according to what the arbitrator likes best (on fairness grounds, say). Then if the arbitrator has a preference for more balanced splitting, an extreme position held by a party need no longer be good because there is a significant risk that the arbitrator will not select it.

Notions of fairness are however likely to be heterogeneous among arbitrators. Arbitrators' view on fairness may in particular depend on their past experiences, personal backgrounds etc...[10] The heterogeneity in arbitrators' preference will in turn translate in randomness in how the parties perceive whether their own position will be selected by the arbitrator. For example in a range of positions in which it is equally likely that the fairness view of the arbitrator dictates that he chooses one position or the other, then parties will perceive that there is a 50/50 chance that their position is selected. To the extent that parties are risk-neutral and arbitrators' fees are constant, this fits again with Example 1 above, and concessions have the effect of increasing one's opponent (expected) arbitration payoff. Of course, this is a highly specific formulation of the heterogeneity of the arbitrators. Under other forms of heterogeneity, the effect of a concession on one's opponent (expected) arbitration payoff may be milder, or possibly negative. Our general analysis however allows us to understand the mapping between the form of arbitrators' heterogeneity and gradualism in the bargaining phase.

The second bargaining interpretation for the termination option is in terms of multi-session negotiation. More precisely, suppose bargaining takes the following form. There are several sessions of negotiation (say one every year). Each session is made of several (finitely many) rounds, say T rounds. After the T^{th} round, the

[10] See for example Bazerman (1985).

session is closed. Items which have been agreed upon during the session are implemented. Items which have not been agreed upon are postponed till the next session (where possibly new items will be added to the agenda), and so on. Readers will recognize a very familiar pattern of negotiation (either inspired from Departmental meetings or more significantly from trade negotiations in WTO or even congress parliamentary sessions).

The above setup does not explicitly include a termination option for the bargaining parties. However, by just waiting till the T^{th} round without making any concession, each party has the option to force the closure of the current session at current terms. In so doing, the party benefits from the concessions made so far within the current session, as items which have been agreed upon during the session are assumed to be implemented. And the negotiation on the unsettled issues will restart in the forthcoming session. Our formulation fits in well with this setup by interpreting the termination outcome as the reduced form for the equilibrium outcome in the subgames where some items are left for future sessions. The complete analysis of this nested session problem is left for future research.

References

[1] Abreu, D. and F. Gul (2000): "Bargaining and Reputation," *Econometrica*, **68**, 85-117.

[2] Admati A.R. and M. Perry (1991), "Joint Projects without Commitment," *Review of Economic Studies* **58**, 259-276.

[3] Bazerman, M.H. (1985), "Norms of Distributive Justice in Interest Arbitration", *Industrial and Labor Relations Review*, 38, 558-570.

[4] Bliss, C. and Barry Nalebuff (1984), "Dragon Slaying and Ballroom Dancing: The Private Supply of a Public Good," *Journal of Public Economics*, 25, 1–12.

[5] Compte O. and P. Jehiel (1995): “On the Role of Arbitration in Negotiations,” mimeo CERAS.

[6] Compte O. and P. Jehiel (2000a): “On the Role of Outside Options in Bargaining with Obstinate Types,” mimeo CERAS and UCL and forthcoming *Econometrica*.

[7] Compte O. and P. Jehiel (2000b): ”Voluntary Contributions to a Joint Project: Revisiting Admati and Perry ’s Contribution Game,” mimeo CERAS and UCL.

[8] Feuille, P. (1975), “Final Offer Arbitration and the Chilling Effect,” *Industrial Relations*, **14**, 302-310.

[9] Gale, D. (2000), “Monotone Games with Positive Spillovers”, mimeo

[10] Kennan, J., and R. Wilson (1993), “Bargaining with Private Information,” *Journal of Economic Literature* **31**, 45-104.

[11] Lockwood B. and J. Thomas (1999): “Gradualism and Irreversibility,” mimeo Warwick University.

[12] Marx L. and S. Matthews (2000): “Dynamic Voluntary Contribution to a Public Project,” *Review of Economic Studies.*

[13] Nash J. (1950): “The Bargaining Problem,” *Econometrica*, **18**, 155- 162.

[14] Rubinstein, A. (1982), “Perfect Equilibrium in a Bargaining Model,” *Econometrica* **50**, 97-109.

[15] Stahl I (1972): “Bargaining Theory,” Stockholm School of Economics

[16] Stevens, C.M. (1966), “Is Compulsory Arbitration compatible with Bargaining?” *Industrial Relations* **5**, 38-50.

Appendix

Gradual equilibrium concessions: an example

Our main result leaves open whether parties will in equilibrium make concessions or eventually decide to opt out. In this Appendix we show that sometimes parties will choose to enter the process of concessions and counterconcessions.

To this end, we consider bargaining game 2, with the assumption that $v_i^{out}(x)$ solely depends on the share of the pie not conceded yet, which we denote X :

$$\begin{aligned} X &= 1 - x_1 - x_2 \\ v_i^{out}(x) &= w^{out}(1 - x_1 - x_2) \end{aligned}$$

A notable feature of this game is that since concessions are immediately consumed (rather than consumed upon reaching and agreement), the payoff relevant information can be reduced to the share X, which greatly simplifies the analysis. (The analysis of bargaining game 1 is more involved, and was performed in Compte-Jehiel 1995.) Finally, note that the analysis we are about to perform applies to contribution game 1 (the sunk contribution case).[11]

Analysis: Suppose $v_1^0 \geq v_2^0$. We define below a class of behavior strategy profiles that specify what parties do whenever it is their turn to move as a function of the share X not conceded yet (and that generate gradual concessions). We will then examine when these strategies constitute a Subgame Perfect Nash Equilibrium.

The class of behavioral strategy profile we consider is characterized by two sequences $X^{(n)}$ and $\bar{X}^{(n)}$, with $0 \leq n \leq N$, satisfying $X^{(0)} = 0$,

$$\begin{aligned} X^{(n)} &< \bar{X}^{(n)} < X^{(n+1)} \text{ for every } n \text{, and} \\ \bar{X}^{(N)} &< 1 < X^{(N+1)}. \end{aligned}$$

The behavioral strategies associated with two such sequences are defined by:

[11]This is because in that game too, the payoff relevant information can be reduced to a one-dimentional parmeter: the size $(K - k)$ of the contribution that is required to complete the project.

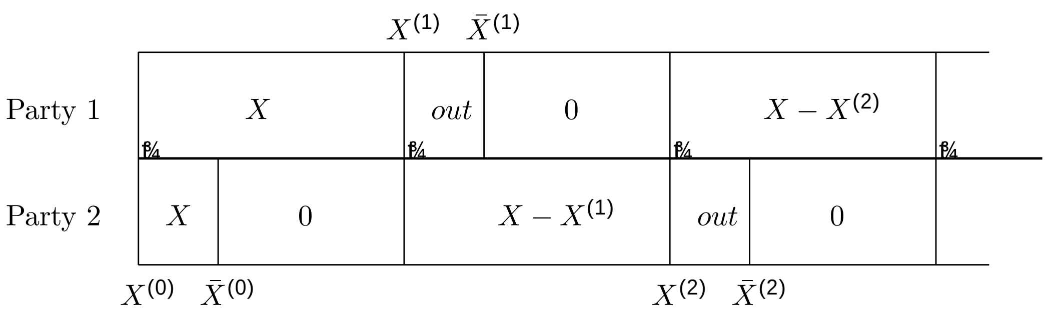
- For X such that $X^{(0)} \leq X \leq X^{(1)}$, party 1 concedes X; party 2 concedes X if $X \leq \overline{X}^{(0)}$, and he concedes 0 otherwise;

And for $n \geq 1$,

- For X such that $X^{(2n-1)} < X \leq X^{(2n)}$, party 2 concedes $X - X^{(2n-1)}$; party 1 opts out if $X \leq \overline{X}^{(2n)}$, and he concedes 0 otherwise.

- For X, $X^{(2n)} < X \leq X^{(2n+1)}$, party 1 concedes $X - X^{(2n)}$; party 2 opts out if $X \leq \overline{X}^{(2n)}$, and he concedes 0 otherwise.

These strategies are summarized in Figure 1.



The axis indicates the share X left to be conceded

We have indicated inside the boxes the equilibrium move of each party as a function of X

Figure 1: Equilibrium with gradual concessions

When parties follow these strategies, parties alternate in making concessions. If for example N is even, party 1 makes an initial concession equal to $1 - X^{(N)}$, next party 2 concedes $X^{(N)} - X^{(N-1)}$, next party 1 concedes $X^{(N-1)} - X^{(N-2)}$, and so on until there is nothing left to be conceded, that is, until $X = X^{(0)} = 0$.

We now exhibit a pair of sequences $X^{(n)}$ and $\bar{X}^{(n)}$ for which the strategies defined above are in equilibrium. The sequence $X^{(n)}$ is defined by induction on n. We choose

$X^{(1)}$ and $X^{(2)}$ such that

$$w^{out}(X^{(1)}) = v_1^0 \text{ and } w^{out}(X^{(2)}) = \delta(X^{(1)} + v_2^0) \tag{11}$$

and for every $n \geq 3$, we choose $X^{(n)}$ such that

$$w^{out}(X^{(n)}) = \delta(X^{(n-1)} - X^{(n-2)}) + \delta^2 w^{out}(X^{(n-2)}). \tag{12}$$

The sequence $\bar{X}^{(n)}$ is defined by $\bar{X}^{(0)} = \frac{1-\delta}{\delta} v_2^0$, and for every $n = 1, 2...$,

$$w^{out}(\bar{X}^{(n)}) = \delta(\bar{X}^{(n)} - X^{(n)}) + \delta^2 w^{out}(X^{(n)}). \tag{13}$$

For δ sufficiently close to 1, these equations define sequences that have the desired properties.[12] To understand how $X^{(n)}$ is constructed, consider for example the case where $X = X^{(n)}$ and n is odd, so that party 1 is supposed to concede a positive share. Expression (12) may be rewritten as

$$w^{out}(X^{(n)}) = \sum_{1 \leq s \leq n/2} \delta^{2s-1}(X^{(n-2s+1)} - X^{(n-2s)}) + \delta^{n-1} v_1^0$$

implying that $w^{out}(X^{(n)})$ is equal to the discounted payoff party 1 obtains if he concedes $X^{(n)} - X^{(n-1)}$ now and the parties behave according to the strategies defined above. In other words, at $X = X^{(n)}$, the party who is supposed to concede, say party 1, is indifferent between opting out and conceding. Note that since this is true for any $n \geq 1$, then (under the proposed strategies), the party who is supposed to concede a strictly positive share does so to a point where the other party is precisely indifferent between opting out and conceding further (and under the proposed strategy that other party concedes). Now to understand how $\overline{X}^{(n)}$ is constructed, observe that

[12] To see why, and to illustrate further the size of the concessions made under this strategy profile, consider the case where $w^{out}(X) = X/2 - \gamma$, and let the extra payoffs v_i^0, $i = 1, 2$ be small and the discount factor δ be close to 1. The last concession (made by party 1) is close to 2γ; the previous concession (made by party 2) is close to 4γ. Rearranging equation (12), we get that $X^{(n)} - X^{(n-1)} \approx X^{(n-1)} - X^{(n-2)}$ for all $n > 2$ so that all previous equilibrium concessions except possibly the first one are close to $X^{(2)} - X^{(1)} \approx 4\gamma$. Besides, when δ is very close to 1, $\overline{X}^{(n)}$ is very close to $X^{(n)}$. So we have $X^{(n)} < \bar{X}^{(n)} < X^{(n+1)}$ as desired.

at $X = \overline{X}^{(n)}$ party 1 is indifferent between opting out and waiting for party 2 to concede $X - X^{(n)}$ in the next stage.

We have the following Proposition:

Proposition 6 *The behavioral strategies associated with the sequences $X^{(n)}$ and $\bar{X}^{(n)}$ defined by (11)-(13) constitute a subgame perfect equilibrium.*

Proof : We check that in every subgames for which the share left is equal to X, no profitable one-shot deviations exist. We start with the case of low values of $X \leq X^{(1)}$, and then proceed by induction on n.

Step 1: $X \leq X^{(1)}$.

Party 2 obtains v_2^0 if she concedes the rest immediately, or $\delta(X - c_2 + v_2^0)$ if she makes a partial concession equal to c_2,with $c_2 < X$, (because party 1 concedes $X - c_2$ in the next period). By definition of $\bar{X}^{(0)} \equiv \frac{1-\delta}{\delta} v_2^0$, it is therefore optimal for party 2 to concede the rest if $X \leq \bar{X}^{(0)}$, and to concede nothing if $X > \bar{X}^{(0)}$.

Party 1 may obtain a share at most equal to $\bar{X}^{(0)}$ from party 2. Since v_1^0 is larger than $\delta(\bar{X}^{(0)} + v_1^0)$, party 1 prefers conceding the rest immediately (the preference is strict if $v_1^0 > v_2^0$).

Step 2: $X^{(n)} < X \leq X^{(n+1)}$, $n \geq 1$.

We let $i = 1$ (respectively $i = 2$) if n is odd, and we let party j be the party other than i. Under the proposed equilibrium strategies, party i (if it is his turn to move) concedes $X - X^{(n)}$ and obtains a payoff equal to $w^{out}(X^{(n+1)})$ (by definition of $X^{(n+1)}$), and party j if it is her turn to move, either opts out (if $X \leq \bar{X}^{(n)}$) or concedes nothing. We will check each party's incentives shortly. We start with two preliminary observations.

a) *It cannot be optimal for a player, say party* 1, *to concede down to a level* X' *from which the other party does not make a counter concession.*

Indeed, if party 2 opts out at X', party 1 obtains $\delta w^{out}(X')$, hence he would have rather opted out right away. And if party 2 concedes nothing and say $X^{(l)} < X' \leq X^{(l+1)}$, party 1 would have rather conceded $X - X^{(l)}$ right away.

b) *It cannot be optimal for a player, say party* 1*, to concede* $X - X'$ *with* $X' < X$ *and* $X^{(l)} < X' < X^{(l+1)}$*.* Indeed, either the above case applies, or party 2 is supposed to concede next, in which case she concedes $X' - X^{(l)}$. But party 1 would rather concede less than $X - X'$ because party 2 would concede down to $X^{(l)}$ in any case.[13]

Party i's incentives: To satisfy a) and b), party i must make a concession equal to $X - X^{(n-2s)}$ for some $s \geq 0$. The resulting payoff for party 1 is equal to $w^{out}(X^{(n-2s+1)})$, and it is largest for $s = 0$.

Party j's incentives: To satisfy a) and b), if party j makes a positive concession, that concession must be equal to $X - X^{(n-2s-1)}$ for some $s \geq 0$. The resulting payoff for party 2 is equal to $w^{out}(X^{(n-2s)})$, it is largest for $s = 0$, but it is still strictly below the payoff $w^{out}(X)$ that party 2 would obtain by opting out right away. It is thus optimal for party j to either concede nothing or opt out. Which of these two options he prefers depends on whether X is above or below $\bar{X}^{(n)}$. ■

It may be shown that when $v_1^0 > v_2^0$ the above strategies constitute in fact the only Subgame Perfect Nash Equilibrium in this setup (the argument makes iterative use of dominance relations).

[13] The only exception is if party 1 concedes to $X' = \bar{X}^{(0)}$, but such a concession gives party 1 a payoff equal e_1 which is stricly below $w^{out}(X)$, for $X > X^{(1)}$.