On the Role of Outside Options in Bargaining with Obstinate Parties*

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Abstract

The presence of obstinate types in bargaining has been shown to alter dramatically the bargaining equilibrium strategies and outcomes. This paper shows that outside options may cancel out the effect of obstinacy in bargaining. When parties have access to stationary outside options, we show that when opting out is preferable to accepting the inflexible demand of the other party, there is a unique Perfect Bayesian Equilibrium in which each party reveals himself as rational as soon as possible. A similar conclusion holds when outside options may only be available at a later date or when only one party has access to an outside option.

Keywords: bargaining, inflexibility, war of attrition, outside options.

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1 Introduction

The main insight of the literature on bargaining pioneered by Rubinstein (1982) is that in a complete information setting equilibrium strategies are (fully) determined by the relative impatience (or waiting costs) of the bargaining parties. In equilibrium there is an immediate agreement; the proposer makes an offer so that the responder is indifferent between accepting the offer and rejecting it given the cost of waiting.

Since then the bargaining literature has broadly speaking been extended in two main directions: allowing for *asymmetric information* between the parties and incorporating *outside options*. Early attempts to introduce asymmetric information in the bargaining context - for example on the impatience of the parties - (see Rubinstein 1985, Fudenberg-Tirole 1983, Admati-Perry 1987, for example) - have had mitigated success in the sense that *many* Perfect Bayesian Equilibria could arise with very different qualitative properties; the multiplicity arises because the moves in the bargaining game may have various signalling effects on the information privately held by the parties (see the survey by Kennan and Wilson 1993 for an account of this literature).

More recently, Myerson (1991) considered another form of psychologically-oriented asymmetric information, which is in line with the crazy types introduced by Kreps-Milgrom-Roberts-Wilson. Specifically, Myerson considered a version of Rubinstein's bargaining model in which with small probability one of the parties is obstinate (or irrational) and insists on obtaining a fixed share θ (typically larger than the one obtained in the complete information equilibrium outcome) of the pie. He showed that when the bargaining frictions are small (the discount factors are close to 1), agreement occurs almost instantaneously at terms close to the ones that prevail when the party is known to be obstinate with probability one.[1] Thus, in Myerson's model the relative degree of impatience of the parties has no impact on how the pie is split.

Abreu-Gul (2000) pursued this approach to analyze the case of two-sided uncertainty: each party i may be an obstinate party who always insists on θ_i and each party is uncertain as to whether the other party is obstinate or rational. They showed that when the bargaining frictions are small there is a unique equilibrium outcome. Moreover the equilibrium is invariant to the details of the bargaining protocol. The game has the following strategic structure. Suppose a party, say party 1, makes a demand other than θ_1. Then he reveals himself as rational. From Myerson (1991) we may infer that party 2 (who is not known to be rational

[1] The result that the outcome is favorable to the party who may be obstinate is in line with those obtained in the literature on reputation, see Fudenberg-Levine 1989, as well as those obtained in the literature on sequential bargaining under incomplete information, see Fudenberg et al. 1985 and Gul et al. 1986.

for sure) gets approximately her demand θ_2. If $\theta_1 + \theta_2 > 1$ the game has the structure of a war of attrition because no party is willing to reveal himself first (this results in delayed agreements in not too asymmetric cases, see Abreu-Gul 2000).

The Myerson-Abreu-Gul approach thus shows that introducing the possibility that parties are obstinate even with small probability has a deep effect on the bargaining equilibrium strategies and on the equilibrium outcome, which are no longer (solely) determined by the relative degree of impatience.

In this paper, we explore the role of outside options in the Myerson-Abreu-Gul setup. Our main insight is that outside options may under mild assumptions completely cancel out the dramatic effect caused by the possibility of obstinacy in bargaining. More precisely, in the unique Perfect Bayesian Equilibrium of the game, parties when rational no longer try to build a reputation for obstinacy. They rather behave as if there were no obstinate type and use their outside option only when their opponent behaves like an obstinate party.

Before we elaborate on our main results, it may be worthwhile recalling the basic insight about outside options in bargaining when information is complete (see Binmore-Shaked-Sutton 1989). Consider a variant of the Rubinstein bargaining model in which each party upon moving may decide to opt out. Suppose further that when a party opts out he gets a payoff that is inferior to the equilibrium payoff he would obtain in the Rubinstein game without outside options. Then there is a unique Subgame Perfect Equilibrium of this game in which the strategies employed by the parties coincide with those of the original Rubinstein game without outside options. In other words, outside options appear to have no effect on the equilibrium bargaining strategies nor on the equilibrium outcomes in a complete information setup.[2]

Our main result - that outside options cancel out the effect of obstinacy - can most easily be illustrated in the following basic setup. Each party i whenever he moves can decide to opt out resulting in payoffs v_i^{out} and v_j^{out} to parties i and j, respectively, where j stands for the party other than i. Party i discounts future payoffs according to δ_i. Each party i may *a priori* be obstinate with probability ε_i and rational with probability $1 - \varepsilon_i$. When party i is obstinate he insists on the share θ_i of the pie. Any inferior offer is rejected. Three (minimal) assumptions are made regarding the outside option payoffs. First we assume that $v_i^{out} < \delta_i v_i^*$ where $v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j}$ is the equilibrium payoff party i would obtain in the complete information version of the bargaining game where party i moves first (observe that there would be no effect of the outside options in a complete information setting, as noted above). Second we assume that $\theta_i > v_i^*$ so that in the Myerson-Abreu-Gul setup, parties try to build

[2]When outside options depend on the history of bargaining, the equilibrium strategies may require the parties to make less and less extreme demands in a gradual fashion (see Compte-Jehiel 1997).

a reputation for obstinacy leading to an equilibrium outcome that differs (substantially) from the Rubinstein outcome. Third we assume that $v_i^{out} > 1 - \theta_j$ so that each party i prefers to opt out rather than accepting the inflexible demand of party j.

The game described above has a unique Perfect Bayesian Equilibrium: each party i whenever rational makes the same proposal as in the complete information setup without obstinate types (i.e. he offers v_i^* for himself and $1 - v_i^* = \delta_j v_j^*$ for party j); the proposal is accepted by party j when rational. If party i (respectively party j) behaves like the obstinate type i (respectively j), party j (respectively party i) opts out. This conclusion also holds if only one party, say party i, may be obstinate (i.e. $\varepsilon_j = 0$).[3]

To see that outside options will sometimes (i.e., in some subgames) have an effect is straightforward. Consider, for example, a subgame at which party i has convinced party j that he is obstinate with probability close to 1: when there are no outside options, party j has no better alternative than giving in to the inflexible demand of party i; with outside options, party j strictly prefers to opt out. Our contribution is to show that the effect of outside options on the equilibrium bargaining strategies is significant even at the start of the game where the probability that the parties are obstinate is small.

We next extend our basic insight in two main directions. First we consider the case where the outside options are not available immediately but only at a later date so that from the early stage point of view, waiting for the outside option is a poor alternative. Second we consider the case where only one party has an outside option. In these two extensions, we show that when the prior probabilities that parties are obstinate are small and the discount factor approaches 1, parties when rational reveal themselves with a probability close to 1, thus resulting in equilibrium outcomes close to the perfect information ones. These extensions show how powerful outside options are at cancelling out the effect of obstinacy in bargaining.

The remainder of the paper is organized as follows. In Section 2 we describe the model without types. In Section 3 we present and analyze the game with obstinate types. In Section 4, we show the effect of outside options. In Section 5, we extend the analysis to the cases where the outside option may be available at a later date and where only one party has access to the outside option. Concluding remarks are gathered in Section 6.

2 The Basic Model

Two risk neutral parties $i = 1, 2$ bargain on the partition of *a pie of size one* which will be partitioned after the negotiation process stops. Each party moves in alternate order. Every

[3] It is not difficult to show that revealing oneself as rational as soon as possible constitutes a Perfect Bayesian Nash Equilibrium. What is somewhat more surprising is that there is no other equilibrium.

other period, party i is the proposer: party i either opts out or makes a *partition offer* to party j, where j stands for the party other than i. Party j may then either accept the offer, opt out or postpone the negotiation till the next period, where it will be her turn to be the proposer. The negotiation stops when a party accepts an offer or opts out. party i discounts future payoffs with the discount factor $\delta_i \in (0, 1)$.

Formally, we assume that party 1 (respectively 2) is the proposer in odd (respectively even) periods. We let (X_i^t, X_j^t) denote the offer made by party i at date $t \geq 1$. A partition offer satisfies:

$$X_i^t + X_j^t = 1.$$

The scalar X_i^t is called party i's demand. If the offer is accepted by party j at time t, then parties i and j's payoffs at time t are given by X_i^t and X_j^t, respectively. If party j chooses to opt out, then parties i and j obtain respectively:

$$(v_i^{out}, v_j^{out})$$

where it is assumed that:

$$v_i^{out} + v_j^{out} < 1.$$

It is well known that when there are no outside options or when these are not too high, and when both parties are known to be rational with probability 1, equilibrium offers are only driven by the parties' relative waiting costs (the proof is analogous to that in Binmore, Shaked and Sutton 1989 and is therefore omitted):

Proposition 1 *(Rubinstein 1982) For each party i let $v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j}$ where j stands for the party other than i. Assume that for each party i, $v_i^{out} < \delta_i v_i^*$. Then there exists a unique Subgame Perfect Nash Equilibrium. When party i is the proposer, he offers the partition $(v_i^*, \delta_j v_j^*)$ to party j. Party j accepts any offer no smaller than $\delta_j v_j^*$ and rejects any offer strictly smaller than $\delta_j v_j^*$.*

The economic intuition underlying the result is that the outside option has no effect on the outcome of the game when both parties are known to be rational because it is never *credible* for either party to opt out.

3 When parties may be obstinate and there is no outside option

In the rest of the paper we will assume that a party can be either rational or obstinate. Obstinate types are modelled as having a mechanical behavior. Specifically we will assume that party i when obstinate always demands the same amount $\theta_i > v_i^*$ for himself where $v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j}$ is the equilibrium payoff that party i gets when both parties are known to be rational with certainty and party i makes the first offer (see above). We will further assume that such an obstinate party systematically rejects any offer that would yield him strictly less than θ_i, and accept any offer larger than or equal to θ_i. The parameter θ_i will be referred to as the *inflexible demand* of party i's obstinate type.

The aim of the paper is to explore the role of outside options in this context. As a benchmark, this Section considers the case where parties have no access to outside options (or equivalently $v_i^{out} < 0$ for $i = 1, 2$). Most of the insights in this Section can be found in Myerson (1991) and Abreu-Gul (2000).

3.1 One-sided uncertainty

We first examine the case where one party, say party i, is known to be rational (with probability 1), and the other party, i.e. party j, is either rational with probability $(1 - \varepsilon)$ or obstinate with probability $\varepsilon > 0$. We wish to characterize the equilibrium outcome obtained in Perfect Bayesian Nash equilibria of the game, and we are especially interested in the case where the prior probability ε is small.

Since the behavior of the obstinate type is mechanical, we only need to determine the behavior of the parties when rational. Furthermore, as soon as party j's behavior does not correspond to that of the obstinate type, party i will infer that he is facing a rational party with probability 1; the ensuing equilibrium behavior is then that governed by the Rubinstein (1982)'s equilibrium strategies or equivalently the strategies shown in Proposition 1. When party j chooses an action different from that of the obstinate type, we say that she has *revealed herself as rational.*

When party j plays the action of the obstinate type, party i may still be uncertain as to whether he is facing an obstinate party or a rational party who behaves as an obstinate party. The Perfect Bayesian equilibrium concept requires that the updated belief of party i is derived from the actual equilibrium strategy of party j according to Bayes' rule. Note that throughout the paper, we allow for mixed strategies both on the proposer's and the responder's sides.

Formally, consider a strategy profile $\sigma^* = (\sigma_i^*, \sigma_j^*)$ and a history $h = (h_i, h_j)$ where h_k is the history of play of party k. We let μ^h be the belief of party i that party j is obstinate given history h. We also let H_j^* denote the set of histories such that the history of play of party j coincides with that of the obstinate party j. We have:[4]

$$\forall h=(h_i,h_j)\in H_j^*,\ \mu^h=\frac{\varepsilon}{\Pr(h_j\mid h_i,\sigma_j^*)},$$

and for $h\notin H_j^*$, $\mu^h=0$, as party j then has revealed herself as rational. This shows how beliefs $\{\mu^h\}_h$ are formed as a function of the strategy profile σ^*. Now, given a belief system $\widehat{\mu}=\{\mu^h\}_h$ derived from σ^*, we can compute for parties i and j (when rational) the expected payoff induced by the strategy profile $\sigma=(\sigma_i,\sigma_j)$ in any subgame starting after history h; we denote by $u_i(\sigma_i,\sigma_j;\widehat{\mu},h)$ and $u_j(\sigma_i,\sigma_j;\widehat{\mu},h)$ these expected payoffs, respectively. The strategy profile σ^* is a Perfect Bayesian equilibrium if for the belief system $\widehat{\mu}$ derived from σ^* by Bayes' rule, after any history h and for any strategies σ_i, σ_j, $u_i(\sigma_i^*,\sigma_j^*;\widehat{\mu},h)\geq u_i(\sigma_i,\sigma_j^*;\widehat{\mu},h)$ and $u_j(\sigma_i^*,\sigma_j^*;\widehat{\mu},h)\geq u_j(\sigma_i^*,\sigma_j;\widehat{\mu},h)$.

The next Proposition is a variant of a result by Myerson (1991); it shows that in any Perfect Bayesian equilibrium, if the initial probability ε that party j is obstinate is not too small relative to $1-\delta_i$ and $1-\delta_j$, then party j's equilibrium payoff is close to his obstinate demand θ_j.

Proposition 2 *(Extension of Myerson 1991, Theorem 8.4.) Let party i be rational with probability 1. Let party j be the obstinate type $\theta_j>v_j^*$ with probability ε, and the rational type with probability $1-\varepsilon$. Let the ratio $\frac{1-\delta_1}{1-\delta_2}$ be fixed and let $\delta=\min(\delta_i,\delta_j)$. In any Perfect Bayesian equilibrium, parties i and j's (expected) equilibrium payoffs (v_i,v_j) satisfy:*[5]

$$\theta_j-\kappa(1-\delta)\leq v_j\leq\theta_j$$
$$1-\theta_j\leq v_i\leq 1-\theta_j+\kappa(1-\delta).$$

where κ is a constant independent of δ (and of the equilibrium considered).

Proof. See Appendix A. ■

The strength of the above Proposition stems from the fact that the scalar κ is set independently of $\delta=\min(\delta_i,\delta_j)$. Therefore, letting δ go to 1 while keeping all other parameters (i.e. ε,θ_j, and $\frac{1-\delta_1}{1-\delta_2}$) fixed, the Proposition shows that party j gets approximately her inflexible demand θ_j. Besides, party i gets approximately $1-\theta_j$, thus revealing that there is no bargaining delay in equilibrium. Contrary to Rubinstein's complete information setup however, the relative impatience of the parties (i.e. $\frac{1-\delta_1}{1-\delta_2}$) has no impact on how the pie is split.

As in the literature on reputation (see, for example, Fudenberg-Levine 1989), the intuition is that party j by mimicking the obstinate type may affect the equilibrium belief of party i

[4] $\Pr(X\mid Y)$ denotes the probability of event X conditional on Y.

[5] Throughout the paper, payoffs should be interpreted as expected payoffs.

about party j's obstinacy. The question is how long it takes for party j to build a reputation for obstinacy. Proposition 2 shows that the time required to build such a reputation is not too long.

Let us provide a brief intuition for Proposition 2 based on Myerson's proof. In a first step, it is shown that party j must reveal herself completely in a finite number[6] of periods (the length of time it takes to reach complete revelation is equivalent to the length of time it takes for party j to build a reputation for obstinacy). The argument is similar to that of Fudenberg and Levine (1989): for party i, there is a cost to rejecting the inflexible offer $1-\theta_j$ right away (the cost of delaying the benefit of the share $1-\theta_j$); if party i chooses to reject the inflexible offer in equilibrium, it must be because he expects that party j will reveal herself with positive probability in a not too distant future. Therefore, the probability that party j reveals herself cannot vanish, and party j must reveal herself in finite time.

Contrary to the case analyzed by Fudenberg and Levine (1989), the above observation does not permit us to conclude because the cost to party i of delaying one more period the benefit of the share $1-\theta_j$ is very small when δ_i is close to 1. So the probability with which party j ought to reveal herself in equilibrium may be very small too. As a consequence, the length of time it takes to reach complete revelation (or equivalently for party j to build a reputation for inflexibility) may be very long (as δ_i gets close to 1).

Going backwards from the final date however, one can see that although the loss from rejecting an obstinate offer is small, the gain from doing so must be small too, because party j has the option to wait for the final date and secure a share equal to θ_j. This implies that in order to compensate for the loss of delaying the benefit of $1-\theta_j$, the probability that party j accepts party i's next offer must be significant (comparable to $\frac{1}{k}$ where k is the number of periods before the final date), which further implies that complete revelation must occur in a relatively small number of periods. In Appendix A we do not reproduce Myerson's proof but rather we derive more accurate characterizations of equilibrium payoffs, which are needed in subsection 6.1 when we analyze delayed outside options. For completeness we also provide a detailed description of a Perfect Bayesian Equilibrium at the end of Appendix A.

3.2 Two-sided uncertainty

We now turn to the case where both parties can be obstinate with positive probability. Each party i is either rational with probability $(1-\varepsilon_i)$ or obstinate with probability $\varepsilon_i > 0$.

As in the one-sided uncertainty case we only need to determine the behavior of the parties when rational. Furthermore, as soon as one party, say party j, does not behave

[6] This number goes to ∞ as δ goes to 1.

like the obstinate type, the other party, i.e. party i, infers that he is facing a rational party with probability 1; the ensuing equilibrium behavior is then that studied in the one-sided uncertainty case (possibly with different beliefs on the obstinate type). When party j chooses an action different from that of the obstinate type, we will still say that she has *revealed herself as rational.*

As long as both parties behave like their respective obstinate types, we are in the two-sided uncertainty case. That is, each party remains uncertain as to whether the other party is obstinate or not. The equilibrium solution concept we employ is still the Perfect Bayesian Equilibrium concept. The beliefs of the parties are derived from the equilibrium strategies according to Bayes' rule. Formally, consider a strategy profile $\sigma^* = (\sigma_i^*, \sigma_j^*)$ and a history $h = (h_i, h_j)$ where h_k is the history of play of party k. We let μ_j^h be the belief (of party i) that party j is obstinate given history h. We denote by $\widetilde{H}$ the set of histories such that neither party has yet revealed himself as rational. For $i = 1, 2$, we let H_i^* denote the set of histories where party i, but not party j, has behaved according to the obstinate type. Finally, we let ξ_i denote the strategy for party i that consists in mimicking the obstinate party i. We have:

$$\text{For } h \in \widetilde{H}, \quad \mu_j^h = \frac{\varepsilon_j}{\Pr\{h \mid \xi_i, \sigma_j^*\}}.$$

For $h = (h_i, h_j) \in H_i^*$, $\mu_j^h = 0$ and $\mu_i^h = \frac{\varepsilon_i}{\Pr(h_i|h_j, \sigma_j^*)}$. For $h \notin \widetilde{H} \cup H_i^* \cup H_j^*$, $\mu_i^h = \mu_j^h = 0$. This defines a belief system $\widehat{\mu} = \{\mu_i^h, \mu_j^h\}_h$ as a function of the strategy profile $\sigma^* = (\sigma_i^*, \sigma_j^*)$. Given such a belief system, denoted by $\widehat{\mu}$, we may compute for parties i and j (when rational) the expected payoff induced by the strategy profile $\sigma = (\sigma_i, \sigma_j)$ after any history h; we denote by $u_i(\sigma_i, \sigma_j; \widehat{\mu}, h)$ and $u_j(\sigma_i, \sigma_j; \widehat{\mu}, h)$ these expected payoffs, respectively. The strategy profile σ^* is a Perfect Bayesian equilibrium if for the belief system $\widehat{\mu}$ derived from σ^* by Bayes' rule, after every history h, and for any strategies σ_i, σ_j, $u_i(\sigma_i^*, \sigma_j^*; \widehat{\mu}, h) \geq u_i(\sigma_i, \sigma_j^*; \widehat{\mu}, h)$ and $u_j(\sigma_i^*, \sigma_j^*; \widehat{\mu}, h) \geq u_j(\sigma_i^*, \sigma_j; \widehat{\mu}, h)$.

The next Proposition analyzes the Perfect Bayesian equilibria of the game. It is a variant of a result by Abreu-Gul (2000). A key observation is that the game has the structure of a war of attrition, since by Proposition 2 each party would prefer the other party to reveal himself first. What the next Proposition makes clear is the equilibrium outcome of this war of attrition as a function of the parameters of the models.

We let

$$a_i \equiv \frac{(1-\delta_i)(1-\theta_j)}{\theta_i - (1-\theta_j)} \text{ and } \phi_i(\mu) = \int_\mu^1 \frac{d\mu}{\mu a_i} \ (= -\frac{1}{a_i} \ln \mu).$$

We have the following Proposition:

Proposition 3 *(Abreu-Gul 2000) Assume that for $i = 1, 2$, party i is the obstinate type θ_i with probability ε_i, and is rational with probability $1 - \varepsilon_i$. Assume that $\phi_i(\varepsilon_j) > \phi_j(\varepsilon_i)$, and* *choose* z *such that* $\phi_i(z) = \phi_j(\varepsilon_i)$,[7] *and* $\pi = 1 - \varepsilon_j/z$. *Let the ratio* $\frac{1-\delta_1}{1-\delta_2}$ *be fixed and let* $\delta = \min(\delta_i, \delta_j)$. *In any Perfect Bayesian equilibrium, the equilibrium payoffs* (v_i, v_j) *of parties* i *and* j *(when rational) satisfy:*

$$\mid v_i - (1-\pi)(1-\theta_j) - \pi\theta_i \mid \leq b(1-\delta)^{1/2} \qquad \textit{and} \qquad \mid v_j - (1-\theta_i) \mid \leq b(1-\delta)^{1/2}$$

where b *is a constant independent of* δ *(and of the equilibrium considered).*

Proof. See Appendix B. ■

Again the strength of the Proposition stems from the fact that the scalar b is set independently of $\delta = \min(\delta_i, \delta_j)$. Therefore, letting δ go to 1 while keeping all other parameters (i.e. ε_i, ε_j, θ_i, θ_j, and $\frac{1-\delta_j}{1-\delta_i}$) fixed, the Proposition gives a good approximation of the equilibrium payoffs. Note that these payoffs depend on $\frac{1-\delta_j}{1-\delta_i}$ only through π. When $\pi < 1$ and δ is sufficiently close to 1, there is some inefficiency (as the equilibrium payoffs of parties i and j add up to less than 1), which takes the form of bargaining delay.

The interpretation for the numbers a_i and $\phi_i(\mu)$ is as follows. The parameter a_i represents the intensity with which party j ought to reveal herself in order to keep party i indifferent between revealing himself and mimicking the obstinate type another period. The number $\phi_i(\mu)$ represents the time necessary for party i's belief (about party j's obstinacy) to reach 1 starting from μ, on the path where neither party reveals himself as rational. The larger $\phi_i(\varepsilon_j)$, the more difficult it is for party j to build a reputation for obstinacy (because it takes a longer time to drive party i's belief to 1). Thus when $\phi_i(\varepsilon_j) > \phi_j(\varepsilon_i)$ it is easier for party i than for party j to build a reputation for obstinacy. The smaller $\phi_i^{-1}\phi_j(\varepsilon_i)$ compared to ε_j, the better for party i.

Comparing Propositions 1 and 3 reveals that the introduction of obstinacy in bargaining may deeply affect the equilibrium outcome. As a matter of fact, Proposition 3 allows us to make several interesting comparative statics as to how the outcome of the war of attrition is driven by the parameters of the model.[8]

To illustrate the potential inefficiency arising in bargaining contexts with obstinate types, consider the symmetric case where $\varepsilon_i = \varepsilon_j$, $\theta_i = \theta_j = \theta > \frac{1}{2}$, and $\delta_i = \delta_j$. Then it can be checked that $\pi = 0$ and therefore each party i gets approximately $1 - \theta$ in equilibrium as $\delta_i = \delta_j$ gets close to 1. In equilibrium parties reveal themselves with a very small probability in each period and there is an expected loss of $2\theta - 1$ induced by the expected delay before

[7] The number z is uniquely defined because ϕ_i is a decreasing function of μ (a_i is positive).

[8] While these comparative statics could in principle be inferred from Abreu-Gul (2000), we believe the earlier version of this paper was the first systematic attempt for the case of patient players in this respect - see also Kambe (1999) for an alternative attempt (in which the prior beliefs are determined by the first offers).

one party accepts to reveal herself as rational. The reason why in the symmetric case, there are significant inefficiencies is that then parties are equally strong (weak), and thus no party is prepared to give in first with a significant probability (whatever the value of ε). The two-sided uncertainty case is also in sharp contrast with the one-sided uncertainty case in which there is no bargaining inefficiency (see Proposition 2).

The comparative statics with respect to the prior probabilities ε_i, ε_j and the relative impatience ratio $\frac{1-\delta_j}{1-\delta_i}$ are relatively intuitive. Party j improves his position when the prior probability ε_j is larger and/or when she is relatively more patient than party i, i.e. when the ratio $\frac{1-\delta_j}{1-\delta_i}$ is smaller. More interesting is the comparative statics with respect to the inflexible demands θ_i, θ_j. If all other parameters are symmetric, i.e. $\varepsilon_i = \varepsilon_j = \varepsilon$, and $\delta_i = \delta_j$, it so happens that it is the party who has the smallest inflexible demand who wins the war of attrition; in other words, the least extremist party wins the war of attrition (in the limit where ε is small, that party manages to get approximately her inflexible demand, i.e. π is close to 1). The intuition for this apparently counter intuitive result is as follows. When the inflexible demand of party j is less extreme than that of party i, it is more costly for party j to give in and therefore party j manages to build a reputation of obstinacy more easily than party i.

4 The role of outside options

The aim of this Section is to show that when the parties have access to suitable outside options, each party when rational now prefers to reveal herself rather than trying to build a reputation for inflexibility. Thus the play of the parties when rational corresponds to that of the game without obstinate types, and therefore outside options have the substantial effect of cancelling out the possibility of obstinacy in bargaining. As for the analysis without outside options, we will consider first the case of one-sided uncertainty and then move to the case of two-sided uncertainty. The solution concept employed is that of Perfect Bayesian equilibrium and the same definitions as the ones proposed in the above section apply as well here.

4.1 One-sided uncertainty

In this subsection we assume that party i is rational with probability 1. Party j is the obstinate type $\theta_j > v_j^* = \frac{1-\delta_i}{1-\delta_i\delta_j}$ with probability ε, and the rational type with probability $1-\varepsilon$. We also assume that[9]

$$(1-\varepsilon)v_i^* + \varepsilon\delta_i v_i^{out} > v_i^{out}. \tag{1}$$

[9]Note that condition (1) is always met whenever $v_i^* > v_i^{out}$ and δ_i is sufficiently close to 1.

That is, party i prefers to get the share $v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j}$ today with probability $(1-\varepsilon)$ and the outside option tomorrow with probability ε rather than opting out immediately. We have:

Proposition 4 *Assume that party i is rational with probability 1 and that party j is the obstinate type $\theta_j > v_j^*$ with probability ε. Assume that $v_i^{out} > 1-\theta_j$ and $v_j^{out} < v_j^*$.*[10] *Under condition (1), there is a unique Perfect Bayesian equilibrium of this game. Let μ_j^h denote the current equilibrium probability that party j is obstinate after history h. Whatever history h, $\mu_j^h \in \{0, \varepsilon, 1\}$. If $\mu_j^h = 0$, both parties behave as in the complete information strategy profile shown in Proposition 1.*
(1) Consider a period t with history h where party i is the proposer.
If $\mu_j^h = \varepsilon$, party i offers $\delta_j v_j^*$ to party j. Party j (if rational) accepts any offer no smaller than $\delta_j v_j^*$, and rejects any offer strictly smaller than $\delta_j v_j^*$.
If $\mu_j^h = 1$, party i opts out.
(2) Consider a period t with history h where party j is the proposer.
If $\mu_j^h \in \{\varepsilon, 1\}$, party j (if rational) offers $\delta_i v_i^*$ to party i. Party i accepts any offer no smaller than $\delta_i v_i^*$; he opts out when he receives the offer $1-\theta_j$, and he rejects any other offer strictly smaller than $\delta_i v_i^*$.

In other words, Proposition 4 establishes that in equilibrium party j (who may be obstinate) reveals herself as rational as soon as possible. Observe that Proposition 4 holds no matter whether the discount factors δ_i, $i = 1, 2$, are close to 1 or not and no matter how small ε is, as long as condition (1) is met.[11] Proposition 4 tells us that in equilibrium party j cannot hope to get more than the payoff v_j^* she obtains in a complete information setting even though party j is potentially obstinate. This result should be contrasted with the result of Proposition 2.

As a matter of fact it is relatively straightforward to show that the strategies introduced in Proposition 4 constitute a Perfect Bayesian equilibrium. If party j plays according to the obstinate type, then party i will believe that he faces an obstinate type with probability 1, since according to the equilibrium behavior, party j when rational should have revealed herself. Since party i has no hope that party j will change her offer (because party j is believed to be obstinate), and since by assumption the outside option yields more to party i than what he would get by accepting party j's inflexible offer, party i will then opt out. But the outcome of that outside option is less favorable to party j than the outcome party

[10] The result would extend to a situation in which party j could be obstinate of several types $\theta_j \in [\underline{\theta}, \overline{\theta}]$ and $v_i^{out} > 1 - \underline{\theta}$.

[11] If condition (1) were not met, then party i would opt out immediately.

j would have obtained by revealing herself (approximately v_j^*), so she should rather have revealed herself.

Proposition 4 is in fact much stronger, since it shows that this is the only Perfect Bayesian equilibrium. We wish to illustrate why this result is not *a priori* straightforward with the following situation. Consider a pie of size 100 and two equally patient parties ($\delta_1 = \delta_2 \simeq 1$). Rubinstein's perfect information equilibrium outcome is thus very close to $50 - 50$. Assume that party j when obstinate demands 90, and that the probability of obstinacy of party j is very small and equal to 0.01. Assume that party i is rational and has an outside option equal to 11.

Suppose that party j demands 90 (so that party i is *a priori* uncertain as to whether he faces the rational or the obstinate type). If party i opts out, he gets 11. If he stays, it might seem that he still has a good chance of getting more than 11, say 30 if he faces the rational party; and if he faces the obstinate type, he can still get 11 (by opting out later). Thus, if party i believes that party j is rational with a probability that is not too small, he may strictly prefer to wait-and-see instead of opting out immediately after party j's demand of 90. Of course, equilibrium requires party i's belief (and hence his perceived chance of getting more than 11) to be consistent with party j's strategy. Proposition 4 reveals that in equilibrium after party j's demand of 90, party i must believe that he faces the obstinate party j with probability 1, and therefore he must perceive that he has no chance of getting more than 10 by waiting. As a result, there cannot be an equilibrium involving wait-and-see strategies of this sort for party i, and party j when rational cannot hope to get more than 50.

The reason why after party j's inflexible demand, party i must believe that he faces the obstinate party j with probability 1 is as follows. For the rational party j to find it optimal to make the inflexible demand it should be that he hopes to get more than in the Rubinstein outcome (otherwise, he would strictly prefer making the Rubinstein offer now). In particular, if $\bar{v}$ denotes the largest offer made by party i to party j in equilibrium, it should be that $\delta_j \bar{v} \geq v_j^*$ so that indeed party j may prefer to wait for party i's offer rather than revealing himself right away. But now suppose that party i makes an offer Y smaller than $\bar{v}$, but yet larger than $[\delta_j]^2 \bar{v}$, (hence larger than $\delta_j v_j^*$). Then party j if rational should accept: he has no chance of getting an offer more generous than $\bar{v}$ in the future (by definition of $\bar{v}$), he will get even less if party i opts out or if he reveals himself in a later period, and there is no chance that party i will ever accept the obstinate offer (because party i would rather opt out). So for party i, making an offer strictly larger than Y would not be optimal, contradicting the assumption that $\bar{v}$ is the largest offer made by party i in equilibrium.

We wish to emphasize here a key difference with the result of Proposition 2 where party i has no outside option, and for which the argument above does not apply. Even when $\delta_j \bar{v} \geq v_j^*$, party i cannot guarantee that by offering $Y > [\delta_j]^2 \bar{v}$, he will get party j to accept his offer. This is because if it were the case, then by rejecting Y party j would convince party 2 that he is obstinate and obtain θ in the next round.

The following argument develops formally this argument to derive an upper bound on party j's equilibrium payoff, from which Proposition 4 follows. The argument is close to that developed by Shaked and Sutton (1984), but note that it is applied (to the best of our knowledge for the first time) to an incomplete information setup.

Proof of Proposition 4: Without loss of generality, let the possibly obstinate party j be party 2. We denote by $\overline{v}_2$ an upperbound on the equilibrium payoff (the rational) party 2 may obtain in any continuation equilibrium in which party 2 moves first, where the upper bound is taken over every possible prior belief about party 2's obstinacy and over any possible perfect Bayesian equilibrium.

Consider at date t a subgame where party 2 moves first. We compute a bound on $\bar{v}_2$.

If party 2 does not offer $(1-\theta_2, \theta_2)$, she reveals herself as rational. The best payoff she may obtain under that event is v_2^*, since then by rejecting the offer party 1 can secure $\delta_1 v_1^*$ in the next round (see Proposition 1) and $v_2^* = 1 - \delta_1 v_1^*$.

If instead party 2 mimics the obstinate type, her offer will be rejected by party 1 (since party 1 strictly prefers opting out rather than accepting the inflexible demand). Then, either party 1 opts out (in which case party 2 gets $v_2^{out} < v_2^*$), or party 1 makes an offer in the next period (at date $t+1$). In the latter case, any offer $X_2 > \delta_2 \overline{v}_2$ would be accepted by (the rational) party 2 with probability 1 (since if party 2 rejected such an offer, she would obtain a payoff at most equal to $\overline{v}_2$ in the subgame starting at $t+2$). Hence in equilibrium, party 1 must offer at most $\delta_2 \overline{v}_2$, and party 2's equilibrium payoff at date t is therefore bounded by $\max\{(\delta_2)^2 \overline{v}_2, v_2^*\}$. We thus obtain:

$$\overline{v}_2 \leq \max\{(\delta_2)^2 \overline{v}_2, v_2^*\}$$

implying that $\overline{v}_2 = v_2^*$, and further that any offer $X_2 > \delta_2 v_2^*$ is accepted in equilibrium by (the rational) party 2.

The rest of the proof is now straightforward. When it is party 2 's turn to make an offer, and if party 2 is rational, any offer other than $(\delta_1 v_1^*, v_2^*)$ would give party 2 a payoff strictly lower than v_2^*. Thus party 2 reveals herself and offers $(\delta_1 v_1^*, v_2^*)$, which is accepted by party 1. Given this behavior of party 2 and given condition (1), the optimal behavior of party 1 is that explained in the Proposition. ■

4.2 Two-sided uncertainty

We now turn to the case where both parties may be obstinate with positive probability. Each party i is either rational with probability $(1-\varepsilon_i)$ or obstinate with probability $\varepsilon_i > 0$. For each party i, we will assume that

$$(1-\varepsilon_j)v_i^* + \varepsilon_j \delta v_i^{out} > v_i^{out}. \tag{2}$$

That is, each party i prefers to get the share $v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j}$ today with probability $(1-\varepsilon_j)$ and use the outside option tomorrow with probability ε_j rather than opt out immediately.

Proposition 5 *Assume that each party i may be the obstinate type θ_i with probability ε_i. Assume further that for each party i, $v_i^{out} > 1-\theta_j$ and $v_i^{out} < v_i^*$.*[12] *Then for any discount factors δ_i, δ_j there is a unique Perfect Bayesian equilibrium of the game. For $j = 1,2$, let μ_j^h denote the current equilibrium probability that party j is obstinate after history h. Whatever h, and for $j = 1,2$, $\mu_j^h \in \{0, \varepsilon_j, 1\}$. The equilibrium strategy of party i (if rational) is that described in Proposition 4 with $\varepsilon = \varepsilon_j$. The equilibrium strategy of party j (if rational) is defined similarly by exchanging the roles of i and j.*

Proposition 5 shows that when the parties have access to an outside option there is no point in building a reputation for inflexibility: in the unique equilibrium, parties reveal themselves as rational as soon as possible. Thus, in the two-sided uncertainty case as well outside options cancel out the effect of obstinacy in bargaining.

Proof of Proposition 5: By proposing $(X_1 = 1 - X_2, X_2)$ with $X_2 > \delta_2 v_2^*$, party 1 reveals himself as rational, and by Proposition 4, party 2 accepts the offer with probability 1 if rational. This implies that party 1 never makes an offer strictly larger than $\delta_2 v_2^*$ in equilibrium. Besides, by choosing an offer X_2 arbitrarily close to $\delta_2 v_2^*$ in the first period, and since $1 - \delta_2 v_2^* = v_1^*$, we obtain that in the first period, party 1 may secure:

$$v_1 = (1-\varepsilon_2)v_1^* + \varepsilon_2 \delta_1 v_1^{out}.$$

Observe now that any offer $X_2 < \delta_2 v_2^*$ is rejected by party 2. (If $X_2 \neq 1-\theta_1$, party 1 has revealed himself as rational, and party 2 may thus secure a payoff equal to $\delta_2 v_2^*$ by revealing herself too in the next period; If $X_2 = 1-\theta_1$, party 2 strictly prefers to opt out rather than accepting X_2.) Moreover, by an argument similar to that developed for party 1 at the start of the proof, we observe that party 2 never makes an offer strictly larger than $\delta_1 v_1^*$ to party

[12] The result would extend to a situation in which each party j could be obstinate of several types $\theta_j \in [\underline{\theta}_j, \overline{\theta}_j]$ and $v_i^{out} > 1 - \underline{\theta}_j$.

1 in equilibrium; thus in any agreement party 1 cannot expect to obtain a share larger than $v_1^* = 1 - \delta_2 v_2^*$. Also, when party 2 is truly obstinate, party 1 gets at most v_1^{out}.

Finally observe that if party 1 offers $X_2 < \delta_2 v_2^*$ in the first period, then an agreement cannot be reached before date 2. So ex ante party 1 gets at most $\delta_1[(1-\varepsilon_2)v_1^* + \varepsilon_2 v_1^{out}]$, which is strictly smaller than v_1. Hence party 1 when rational strictly prefers to offer $(v_1^*, \delta_2 v_2^*)$ right away. Furthermore, when a party fails to reveal herself, he is believed to be obstinate with probability 1, and the other party if rational opts out as soon as possible. ■

5 Extensions

The above analysis suggests that suitable outside options may cancel out the effect of obstinacy in bargaining. We wish to explore the robustness of this insight in two directions. The first direction concerns the case of delayed outside options. Given the above analysis, it should be clear that close enough to the time where the outside options are available the effect of obstinacy will be cancelled out. However, it is unclear what the effect of the outside options is on the early stage of the game. We will show that parties when rational continue to reveal themselves with a probability close to 1 as soon as possible even when outside options are delayed.

The second direction concerns the case where only one party, say party i, has an outside option. From the previous analysis, one might infer that the effect of party i's outside option is to cancel out the effect of party j's obstinacy, thus resulting in an outcome very favorable to party i (as in Proposition 2). It turns out however, that even in this case, both parties will continue to reveal themselves as rational as soon as possible with a probability close to 1. Thus even one-sided outside options may cancel out the effect of obstinacy on both sides.

To simplify the exposition, we will present the results for the case where the two parties have the same discount factor $\delta_i = \delta_j = \delta$ so that $v_i^* = v_i^* = v^* = \frac{1}{1+\delta}$ ($\simeq \frac{1}{2}$ for δ close to 1), and where the inflexible demands of both parties coincide $\theta_i = \theta_j = \theta$ as well as the prior probabilities $\varepsilon_i = \varepsilon_j = \varepsilon$ in the two-sided uncertainty case.

5.1 Delayed but large vs. immediate but mediocre

In this Subsection, we will assume that the outside option is not available before some date T. At the time where the outside option is available we still assume though that it yields a payoff v_i^{out} larger than $1 - \theta$. The discounted value of the outside option is therefore equal to $\delta^T v_i^{out}$. If $\delta^T v_i^{out}$ is larger than $1 - \theta$, the previous analysis applies showing that the parties reveal themselves with probability 1 as soon as possible. We are more interested here in the case where $\delta^T v_i^{out}$ is smaller than $1-\theta$. Then revealing oneself as rational with probability 1 as soon as possible cannot be part of a Perfect Bayesian Equilibrium. To see this suppose by contradiction that revealing herself as rational for party j is part of a Perfect Bayesian Equilibrium. If instead of revealing herself party j mimicked the obstinate type, then party i would become convinced that he faces the obstinate type with probability 1; Party i could then wait for the outside option to be available, but by assumption this is worse than accepting the inflexible demand of party j. Thus party i when rational would accept the inflexible demand of party j and party j would obtain an expected payoff close to θ (as the probability that party i is obstinate is small), which is more than the supposed equilibrium outcome obtained by revealing herself, i.e. v^* at most.

Thus revealing oneself with probability 1 as soon as possible cannot be part of a Perfect Bayesian Equilibrium whenever $\delta^T v_i^{out} < 1-\theta$ for $i = 1, 2$. On the other hand, the parties cannot mimic the obstinate behavior for ever because there is a point where the outside option is sufficiently near in time to induce the parties to reveal themselves as rational (a time t where $\delta^{T-t} v_i^{out} > 1-\theta$).

In what follows we will keep $\delta^T v_i^{out}$ fixed and let the discount factor go to 1. The interpretation of keeping $\delta^T v_i^{out}$ fixed while letting δ go to 1 is that the outside option is available at a given real time in the future but the frequency between offers is increased to infinity.

We will establish that, for $\delta^T v_i^{out}$ fixed, in equilibrium the parties reveal themselves as soon as possible with a probability close to $1 - O(\varepsilon)$ as the discount factor δ approaches 1.[13] When the ex ante probability of being obstinate ε is small, the resulting effect of the delayed outside option is again to cancel out the effect of the obstinate types.

We start our analysis with the one-sided uncertainty case:

Proposition 6 *Assume that party i is known to be rational and that party j is the obstinate type θ with probability ε. Also assume that at date T, party i has access to an outside option such that $v_i^{out} > 1-\theta$. We let $\underline{v}_i = \delta^T v_i^{out}$ be fixed and let δ go to 1. We define μ^* such that*

$$\frac{1}{\alpha} \mid \ln \mu^* \mid = \mid \ln \frac{\underline{v}_i}{1-\theta} \mid \text{ where } \alpha = \frac{1-\theta}{1/2-(1-\theta)} \tag{3}$$

If $\varepsilon < \mu^* + O(1-\delta)$, then in equilibrium, party i when it is his turn to move first offers δv^* to party j (and v^* for himself) and party j accepts with probability $1 - \varepsilon/\mu^* + O(1-\delta)$.

Proof. See Appendices B, C and D ■.

Proposition 6 asserts that if the probability ε that party j may be obstinate is small relative to μ^* (which is defined independently of ε), then in equilibrium party j reveals

[13]For any x, $O(x)$ denotes a number that has the same order of magnitude as x, that is, there exists a constant A independent of δ and of the equilibrium considered such that $\mid O(x) \mid \leq Ax$.

herself as rational with a probability close to 1 by accepting the partition offer $(v^*, \delta v^*)$ made by party i.

To get some intuition for the result, it will be convenient to let $i = 1$ be the party who is rational for sure and to let τ denote the earliest date t for which $\delta^{T-t} v_1^{out} > 1 - \theta$; that is, the earliest date for which the discounted value of the outside option strictly dominates the value from conceding to the obstinate demand. Note that since $\delta^T v_1^{out} = \underline{v}_1$, and since $-\ln \delta \approx 1 - \delta$, the date τ satisfies:

$$\tau \approx \frac{1}{1-\delta} \mid \ln \frac{\underline{v}_1}{1-\theta} \mid . \tag{4}$$

The proof of Proposition 6 shows that the only offers made in equilibrium are either the Rubinstein partition $(v^*, \delta v^*)$ or partition offers that are close to the one accepted by the obstinate party 2 (i.e. $(1-\theta, \theta)$).[14] The game before date τ has the structure of a *war of attrition* in which if party 1 gives in, payoffs to parties 1 and 2 are approximately $(1-\theta, \theta)$ and, if party 2 gives in, payoffs are approximately $(\frac{1}{2}, \frac{1}{2})$ (for δ close to 1 so that $v^* \simeq \frac{1}{2}$). This war of attrition lasts until date τ (where τ is the first date at which it is credible for party 1 to wait for the outside option). The probability $1 - \frac{\varepsilon}{\mu^*}$ with which party 2 gives in initially is adjusted so that indeed the war of attrition ends exactly at date τ.[15]

The complete argument for the proof of Proposition 6 is in the Appendix. It shows formally why the game before τ has a structure of a war of attrition. A difficulty is that party i (who has no obstinate type to mimic) might *a priori* make offers other than the Rubinstein offer and yet different from that accepted by the obstinate party j. By increasing his offer above the Rubinstein offer, party i might hope to increase the probability that party j accepts. It turns out though that in order to generate an increase in the probability that party j accepts his offer, party i has to offer a share close to θ.[16] This is why only two offers may prevail in equilibrium: either the Rubinstein offer, or offers close to the obstinate partition $(1-\theta, \theta)$; any offer in between has no effect on the probability that party j accepts party i's offer.

Note that in the complete information case, the possibility of intermediate offers deeply

[14] All such offers approach $(1-\theta, \theta)$ as δ converges to 1, see Appendix.

[15] Observe that in such a war of attrition, the time it takes for party 1 to get convinced that he faces an obstinate party 2 when his initial belief is μ is equal to $\frac{1}{\alpha(1-\delta)} \mid \ln \mu \mid$. Together with equation (4), this yields equation (3).

[16] The reason why offers below θ do not increase much the probability of acceptance is that otherwise in the ensuing continuation game the outside option would not be credible any longer, and therefore the equilibrium outcome would be that governed by Proposition 2 (resulting in a payoff close to θ to party j, thus making the acceptance of an offer below θ suboptimal).

affects the structure of the game and forces a unique equilibrium offer[17]. The logic of such a result is that an offer slightly above the one that renders party j indifferent between accepting and delaying the agreement for one more period would be accepted with probability 1 by party j. In our incomplete information setup, this logic does not apply, and the possibility of intermediate offers has little effect.[18]

Proposition 6 has an important implication for the two-sided uncertainty case. When both parties may be obstinate, each party has the option to reveal himself as rational right away. By Proposition 6, we know that the party who reveals himself then gets a payoff approximately equal to v^* (as long as the probability that the other party is obstinate is small). We use that property to derive the following result:

Proposition 7 *Assume that each party may be the obstinate type θ with probability ε. Also assume that at date T, each party i has access to an outside option such that $v_i^{out} > 1-\theta$. Then in any equilibrium, both parties obtain a payoff no smaller than $v^* + O[(\varepsilon/\mu^*)^{1/2}] + O(1-\delta)$, where $\mu^* = \min(\mu_1^*, \mu_2^*)$, and μ_i^* is as defined in Proposition 6.*

Proof. A corollary of Proposition 6 is that by revealing himself right away, party 1 can secure $v^* + O(\varepsilon/\mu^*) + O(1-\delta)$, which is no smaller than $v^* + O[(\varepsilon/\mu^*)^{1/2}] + O(1-\delta)$. Deriving a lower bound on what party 2 can secure is a little less straightforward.

Assume that in equilibrium, party 1 reveals himself with probability p in the first period. When party 1 reveals himself as rational, party 2 gets a payoff equal to $v^* + O(1-\delta)$. If $p \geq 1 - (\varepsilon/\mu^*)^{1/2}$, we obtain the desired result that ex ante party 2 gets a payoff no smaller than $v^* + O[(\varepsilon/\mu^*)^{1/2}] + O(1-\delta)$.

Otherwise $p < 1 - (\varepsilon/\mu^*)^{1/2}$; then, in the event where party 1 mimics the obstinate type, the date 2 updated belief about the obstinacy of party 1 is equal to $\mu = \frac{\varepsilon}{1-p} < (\varepsilon\mu^*)^{1/2}$. By revealing herself at date 2, party 2 can secure $v^* + O(\mu/\mu^*) + O(1-\delta)$ (see Proposition 6), which is no smaller than $v^* + O[(\varepsilon/\mu^*)^{1/2}] + O(1-\delta)$ (since $\mu < (\varepsilon\mu^*)^{1/2}$). Combining this with the payoff obtained by party 2 when party 1 reveals himself in the first period, we thus obtain the wished lower bound on party 2's payoff. ■

Some interesting implications may be inferred from Proposition 7. Under the conditions of Proposition 7, in the absence of outside options, a share close to $2\theta - 1$ is lost in delays as a result of a war of attrition. Proposition 7 shows that even delayed outside options are

[17] When there are only two possible offers there are multiple equilibria (see for example van Damme et al. 1990). With intermediate offers, as shown by Rubinstein there is a unique equilibrium.

[18] The reason is that if party j whenever rational were to accept with probability 1 an offer other than the obstinate partition (and significantly smaller than the obstinate demand), by rejecting the offer he would be considered to be obstinate with probability 1, therefore making the first acceptance suboptimal.

quite effective at eliminating the inefficiencies arising in negotiation contexts with obstinate behaviors provided that at the time where the outside option is available it yields a payoff no smaller than the one resulting from accepting the inflexible demand of the obstinate type. Note that this result should be contrasted with the one we would have obtained if the outside option had been stationary, yielding payoffs below $1-\theta$: Then the analysis of Proposition 3 would have applied, resulting in an efficiency loss of approximately $2\theta-1$. (This is so because opting out would always have been dominated by accepting the inflexible behavior of the other party.) We may infer from this consideration that from an efficiency viewpoint it is preferable that outside options be delayed but large, rather than immediately available but mediocre.

As an illustration, consider a bargaining context in which parties may search for outside options, for example by forming a new match with another partner. In general, searching for an outside option requires some time, e.g. T periods, and is costly relative to the perfect information equilibrium outcome of the bargaining with the current partner, i.e. $v_i^{out} < \frac{\delta}{1+\delta}$ for $i = 1, 2$.[19]

A question of interest is whether or not one should let the current partners look for outside options while bargaining. Our theory suggests a positive role for letting the parties do so. To the extent that for $i = 1, 2$, $v_i^{out} > 1-\theta$, but $\delta^T v_i^{out} < 1-\theta$, where θ is the inflexible demand of the other party j, it is preferable that each party starts looking for outside options at the start of the bargaining, in period 1. In so doing, the bargaining takes the form studied in this Section resulting in an efficient outcome with a large probability. On the other hand, if parties must stop bargaining before they start searching for an outside option, at least T periods after the decision of closing the negotiation will be required before that outside option is available. The game has then the structure of a bargaining with stationary and mediocre outside options so that a very inefficient outcome may arise (see above).

In short, letting the parties look for outside options while bargaining makes it less rewarding for the parties to build a reputation for obstinacy, which in turn is beneficial to the bargaining outcome.[20]

[19] This implicitly assumes that after an adjustment cost (already borne in the current match), all matches generate approximately the same surplus.

[20] An argument against letting the parties look for outside options while bargaining is that it may adversely affect the incentives of the parties to do effort to increase the pie (see Frankel 1998 for such a model). A more complete theory should combine this moral hazard argument with the insight provided in this Section.

5.2 One-sided outside options

In this Subsection, we analyze the Perfect Bayesian equilibria of the game when only one party, say party i, has access to an outside option. In cases where at least one party is known to be rational, one of the above Propositions applies. If party j is known to be rational, Proposition 2 applies (since party j has no outside option, party i is able to get approximately (as δ goes to 1) his inflexible demand θ in any Perfect Bayesian equilibrium). When party i is known to be rational, Proposition 4 applies (we did not use the fact that party j had an outside option in the proof).

It thus remains to analyze the case where both parties may be the obstinate type θ.

Proposition 8 *Assume that each party may be the obstinate type θ with probability ε. Assume that only party i has access to an outside option and that $v_i^{out} > 1 - \theta$. Then in any Perfect Bayesian equilibrium both parties obtain a payoff no smaller than:*

$$v^* + O(\varepsilon^c) + O((1-\delta)^{1/2})$$

for some constant $c > 0$ (that is independent of ε).

Proof. See Appendix B ■.

The above Proposition shows that the outside option of party i turns out to have a greater impact than simply cancelling out the effect of the inflexibility of party j; it cancels out the effect of the inflexibility of both parties.

We now provide some intuition for this result. The game has the structure of a war of attrition where the rational parties' payoffs are approximately given by[21]

$$(\theta, 1-\theta)$$

if party j reveals himself first (this follows from Proposition 2) and by

$$((1-\mu^t)v^* + \mu^t \delta v_i^{out}, \delta v^*)$$

if party i reveals himself first and μ^t denotes the current period t belief that party j is obstinate (this follows from Proposition 4). The next step is to observe that because the cost of revealing oneself is much higher for party j than for party i, it is much easier for party j to build a reputation for obstinacy. As a result in equilibrium party i will reveal himself with a probability close to 1, and equilibrium payoffs will be close to $\frac{1}{2}$.

[21] The first (resp. second) scalar indicates party i (resp. j)'s payoff.

6 Conclusion

The main message of this paper is that outside options are quite effective at cancelling out the effect of obstinate types in bargaining. This message has been shown to be robust in several directions: In particular we have shown that even if the outside option is to be delivered at a later date, it induces the parties not to mimic an obstinate behavior provided that at the time where the outside option is delivered it is a better alternative than the one resulting from accepting the inflexible demand of the other party when obstinate. As mentioned in subsection 5.1, this gives some argument in favor of letting the parties look for outside options while bargaining.

Several directions for future research are worth mentioning. We have modelled obstinate behaviors here as ones where a party always insists on the same partition. Sometimes, obstinate behaviors are better viewed as ones that change over time, for example, as a reaction to the behavior of the opponent. It would clearly be of considerable interest to extend the above analysis to dynamic specifications of obstinate behaviors fitting with empirical observations about the actual behavior of the negotiating parties. Also, once we allow for dynamic specifications of the obstinate behavior it may be of interest to analyze how the outside options should depend on the behavior of the parties during the negotiation to be more effective at eliminating the inefficiencies.

Negotiations with arbitration clauses constitute an interesting institutional setting of negotiations with outside options. In the context of arbitration it is often the case that the arbitrator has access to the negotiation hearings. What is the best way for the arbitrator to take into account those hearings in order to reduce the scope for reputation building in the negotiation phase? From the above analysis, it seems that awarding each party with at least the most generous offer of the other party is a good way to combat the perverse effect of reputation building, as this gives (in general) each party an option that dominates acceptance of the current offer made by the other party (who may be following an obstinate behavior). Further work is clearly required to elaborate on these ideas.

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Appendix A

This Appendix is devoted to the analysis of the game where only one party, say party 2, may be obstinate and no outside options are available. We compute equilibrium payoffs and obtain Proposition 2 as a Corollary. For completeness, we will also exhibit an equilibrium strategy profile of that game.

Let us mention that our analysis bears some similarities with that of the durable good monopoly. For party 2 (the party who may be obstinate), making an offer different from θ is a very costly move, as it leads party 2 to reveal herself. And it will turn out that party 2 never makes an offer different from θ in equilibrium (under the conditions of Proposition 2, for patient enough parties). In equilibrium, strategic behavior for party 2 will thus consist solely of acceptance/rejection decisions, while party 1, by his choice of offers, will attempt to get the rational party 2 to accept offers below θ. And as in the durable good monopoly case, the problem of party 1 is that he cannot commit not to make more favorable offers in the future.

We start by computing equilibrium payoffs in subgames where party 1's belief about party 2's obstinacy is high. Let $v^0 = \delta_2\theta$, define the function

$$w^0(\mu) = \max\{(1-\mu)(1-v^0) + \mu\delta_1(1-\theta), 1-\theta\},$$

and set μ^0 so that

$$\delta_1 w^0(\mu^0) = 1-\theta.$$

We first prove the following Proposition, which derives the solution for beliefs above μ^0:

Proposition 9 *For any $\mu \geq \mu^0$, party 1's equilibrium payoff is uniquely defined and equal to $w^0(\mu)$. Besides, if $\mu \geq \mu^0$, party 1 offers at least v^0 to party 2 in equilibrium, and if $\mu < \mu^0$, party 1 offers at most v^0 to party 2 in equilibrium.*

Proof. Party 1 obtains $1-\theta$ when he offers θ to party 2. Also, any offer $Y \in (v^0, \theta)$ would be accepted by the rational party 2 (because he can expect at most θ in the following period) and rejected by the obstinate party (in which case party 1 accepts $1-\theta$ in the following period). Party 1 may thus obtain $(1-\mu)(1-Y) + \mu\delta_1(1-\theta)$, which is largest when Y tends to v^0. This shows that i) party 1's equilibrium payoff is at least equal to $w^0(\mu)$; ii) no offer $Y > v^0$ may be optimal when $\mu < \mu^0$ (since $1-\theta < w^0(\mu)$).

When party 1 makes an offer $Y < \theta$, the best he can expect is that the rational party 2 accepts with probability 1, implying that $w^0(\mu)$ is the largest payoff party 1 can expect when he offers $Y \geq v^0$.

Now assume that the Proposition holds for $\mu \geq \mu^* > \mu^0$. Consider $\mu \in [\xi\mu^*, \mu^*)$, with $\xi\mu^* \geq \mu^0$ and ξ close to 1. If party 1 offers $Y < v^0$, and if party 2 accepts with probability at least equal to $1 - \mu/\mu^*$, then in the next period, party 1 would obtain at most $\delta_1 w^0(\mu^*)$ by rejecting party 2's obstinate demand, which is strictly below $1-\theta$ because $\mu^* > \mu^0$. So party 1 would prefer to accept. Hence party 2 if rational would prefer to reject $Y < v^0$ and get θ in the following period (contradiction). So any offer $Y < v^0$ is accepted with probability strictly smaller than $1 - \mu/\mu^*$. If $t \geq 2$ is the first date t where party 1 makes an offer $Y \geq v^0$, then party 1 obtains a payoff at most equal to

$$(1 - \mu/\mu^*) + \mu/\mu^* \max\{\delta_1(1-\theta), [\delta_1]^t w^0(\mu)\}$$

which is strictly smaller than $1 - \theta$ when $\xi \geq 1 - (1-\theta)(1-\delta)$. Therefore offering $Y < v^0$ is not optimal and party 1's equilibrium payoff is equal to $w^0(\mu)$. ■

Before extending this Proposition to smaller beliefs, we would like to make an important observation. It is easy to check that

$$\mu^0 \leq 1 - \frac{2(1-\theta)}{1+(r-1)\theta}, \text{ where } r = \frac{1-\delta_2}{1-\delta_1}.$$

So for a fixed r, the critical belief μ^0 is bounded away from 1, even when parties are very patient. The reason is as follows. Party 1 may induce the rational party 2 to accept an offer different from θ, by offering $v^0 = \delta_2\theta$ to party 2. However, when δ_2 is close to 1, party 1 cannot gain much from doing so, because v^0 is close to θ. So party 1 will only be willing to do so when the probability that party 2 accepts (hence when the probability that party 2 is rational) is significant.

The observation above is important, because a similar insight holds even when the current belief is smaller μ^0. This will give a lower bound on party 2's acceptance probability, showing that the rational party 2 may build up a reputation for obstinacy very fast. More precisely, define for all $n \geq 0$

$$v^n = [\delta_2]^{2n+1}\theta. \tag{5}$$

A typical equilibrium sequence of offers made by party 1 to party 2 is, as it will turn out, $v^k, v^{k-1}, ..., v^0$ [because such sequences make party 2 indifferent between accepting and rejecting the offer made by party 1.[22]]. Party 1 however does not gain much from having v^k being accepted, rather than v^{k-1}. So inducing party 1 to make an offer v^k now rather than v^{k-1} will require that party 2 accepts v^k with significant probability.

[22]If at one date, party 2 had strict incentives to accept some offer $v < v^0$, then by rejecting the offer, party 2 would convince party 1 that he is obstinate, and therefore obatin θ in the next period. Since $v < v^0 = \delta_2\theta$, this would contradict the hypothesis that party 2 had strict incentives to accept v.

The following notations will allow us to characterize and obtain a lower bound on these acceptance probabilities. Let $\rho = \frac{1-[\delta_2]^2}{1-[\delta_1]^2}$, $\pi^0 = 1-\mu^0$ and $w^0 = w^0(\mu^0)$. Also let N be largest integer for which

$$[\delta_2]^{2N}\theta > v_2^*. \tag{6}$$

Consider the sequence $\{\pi^n, \mu^n, w^n, w^n(.)\}_{0\leq n\leq N}$ defined recursively by:

$$\pi^{n+1} = \frac{w^n}{w^n + \rho v^n}\ , \tag{7}$$

$$\mu^{n+1} = \prod_{k\leq n+1}(1-\pi^k), \tag{8}$$

$$w^{n+1} = \pi^{n+1}(1+(\rho-1)v^n), \tag{9}$$

$$w^{n+1}(\mu) = (1-\mu/\mu^n)(1-v^{n+1}) + (\mu/\mu^n)[\delta_1]^2 w^n. \tag{10}$$

As we show shortly (see Lemma 1), the sequences defined above have been chosen to ensure that for all $n \in \{0, ..., N-1\}$,

$$w^{n+1} = \pi^{n+1}(1-v^{n+1}) + (1-\pi^{n+1})[\delta_1]^2 w^n, \tag{11}$$

or equivalently,

$$w^n = w^n(\mu^n) = w^{n-1}(\mu^n) \tag{12}$$

As mentioned before, a typical equilibrium sequence of offers by party 1 is $v^k, v^{k-1}, ..., v^0$. Along such a sequence, as it will turn out, party 2 accepts v^k with probability π^k; the belief μ^{k-1} then corresponds to that held by party 1 in the event where v^k is rejected; and the value w^{k-1} corresponds to the continuation value of party 1 in the subgame where he makes an offer and v^k has just been rejected. Also, for any $\mu \leq \mu^{n-1}$, the value $w^n(\mu)$ may then be interpreted as the payoff obtained by party 1 when he offers v^n and party 2 accepts with probability $(1-\mu/\mu^{n-1})$.

Lemma 1 *Let* $\underline{\pi} \equiv \frac{1-\theta}{1-\theta+\theta\rho}$. *For all* $n \in \{0, ..., N-1\}$, *(12) holds,* $w^{n+1} > w^n$ *and* $\pi^{n+1} \geq \underline{\pi}$.

Proof. Assume $w^n(\mu^n) = w^n$. First rewrite (10) as

$$w^n(\mu) - 1 + v^n = \frac{\mu}{\mu^{n-1}}([\delta_1]^2 w^{n-1} + v^n - 1) \tag{13}$$

We have

$$w^n(\mu^{n+1}) - w^n(\mu^n) = \frac{\mu^{n+1}-\mu^n}{\mu^n}[\frac{\mu^n}{\mu^{n-1}}([\delta_1]^2 w^{n-1} + v^n - 1)]$$

which, combined with (13) implies[23]

$$w^n(\mu^{n+1}) - w^n(\mu^n) = \pi^{n+1}[1 - v^n - w^n(\mu^n)] \tag{14}$$

[23]Note that Equation (14) holds for $n = 0$ as well because $w^0(\mu^0) = w^0$ and because for $\mu \leq \mu^0$, $w^0(\mu^1) = \pi^1[1-v^n] + (1-\pi^1)w^0$.

Applying (10), we also have, for all $n \geq 0$,

$$w^{n+1}(\mu^{n+1}) - w^n = \pi^{n+1}(1 - v^{n+1} - [\delta_1]^2 w^n) + [\delta_1]^2 w^n - w^n \tag{15}$$

By definition of π^{n+1}, we have, for all $n \geq 0$

$$w^n(1 - [\delta_1]^2) = \pi^{n+1}(w^n(1 - [\delta_1]^2) + v^n(1 - [\delta_2]^2). \tag{16}$$

Since $w^n(\mu^n) = w^n$, and since $v^{n+1} = [\delta_2]^2 v^n$, combining (14), (15) and (16) yields $w^{n+1}(\mu^{n+1}) = w^n(\mu^{n+1})$. Also, combining (14) and (16) yields

$$w^n(\mu^{n+1}) = \pi^{n+1}[1 - v^n - w^n + w^n + \rho v^n] = w^{n+1}.$$

Since $w^0(\mu^0) = w^0$, we conclude (by induction on n) that $w^n(\mu^{n+1}) = w^{n+1}(\mu^{n+1}) = w^{n+1}$ for all $n \geq 0$. Finally, observe that Equation (13) implies

$$1 - v^n - w^n \geq (1 - \pi^n)(1 - v^{n-1} - [\delta_1]^2 w^{n-1}).$$

Since $1 - v^n - [\delta_1]^2 w^n \geq 1 - v^n - w^n$, and since $1 - v^0 - [\delta_1]^2 w^0 > 0$, we obtain $1 - v^n - w^n > 0$ for all $n \geq 0$, which further implies, using (14), $w^{n+1} > w^n$, as desired. It follows that $w^n > 1 - \theta$ for all $n \geq 0$, which implies $\pi^{n+1} \geq \underline{\pi}$ by (7). ■

The following Proposition gives properties of any Perfect Bayesian equilibrium of the game without outside options in which party 2 is believed to be obstinate with probability μ.

Proposition 10 *Define N and $\{\mu^n, v^n, w^n(.)\}_{0 \leq n \leq N}$ as in (5-10). Let $w(\mu)$ be the function that coincides with $w^n(\mu)$ on each interval $(\mu^{n+1}, \mu^n]$, $n \in \{0, .., N\}$ (set $\mu^{N+1} = 0$). Consider a date where party 1 moves first and where party 2 is believed to be obstinate with probability $\mu > \mu^N$. party 1's equilibrium payoff is uniquely defined and equal to $w(\mu)$. Besides, in equilibrium, when $\mu \leq \mu^0$: a) party 1 offers v^n for some n, b) if $\mu > \mu^{n+1}$, then party 1 offers at least v^n, and if $\mu < \mu^n$, party 1 offers at most v^n. c) If $\mu < \mu^n$, party 2's equilibrium payoff is at most equal to v^n.*

Note that Property c) above is redundant, as it is an immediate corollary of Property b). It is included to simplify exposition of the proof.

We first observe that Proposition 2 is a simple Corollary of Proposition 10.

Proof of Proposition 2: Define $\underline{\pi}$ as in Lemma 1 and choose n^* such that $(1 - \underline{\pi})^{n^*} < \varepsilon$. Since the ratio $\frac{1-\delta_2}{1-\delta_1}$ is fixed and since $\rho \geq \frac{1}{2}\frac{1-\delta_2}{1-\delta_1}$, the number n^* has an upper bound independent of δ, so for δ close enough to 1, $[\delta_2]^{2n^*}\theta > v_2^*$, hence $\mu^{n^*} > \mu^N$. Besides, it immediately follows from Lemma 1 that $\mu^{n^*} \leq (1 - \underline{\pi})^{n^*} < \varepsilon$. Thus, by Proposition 10, at any date where he moves, party 1 offers at least v^{n^*-1} to party 2. So party 2's equilibrium payoff is at least equal to $[\delta_2]^{2n^*-1}\theta$, and party 1's equilibrium payoff is at most equal to $1 - v^{n^*-1} \leq 1 - [\delta_2]^{2n^*-1}\theta$, which gives us the desired result (since n^* has an upper bound independent of δ). ■

Proof of Proposition 10: Choose $\xi < 1$ such that

$$1 - \xi < \max\{(1 - (\delta_1)^2)(1 - \theta), \underline{\pi}\}.$$

We will show that if Proposition 10 holds for $\mu \geq \mu^*$, then it also holds for $\mu \geq \xi\mu^*$.[24] In what follows, we assume $\mu^* \in (\mu^{n+1}, \mu^n]$. We start with some notation. We let $\bar{v}_2$ denote an upperbound on party 2's equilibrium payoff when μ belongs to $[\xi\mu^*, \mu^*)$:

$$\bar{v}_2 = \sup\{v_2(\sigma), \mu \in [\xi\mu^*, \mu^*), \sigma \in \Sigma^*_\mu\},$$

where Σ^*_μ denotes the set of equilibria of the game without outside options where party 2 is initially obstinate with probability μ. Note that since $\mu < \mu^* \leq \mu^0$, we already know from Proposition 9 that $\bar{v}_2 \leq v^0$. One of our objective will be to provide a better bound on $\bar{v}_2$. For any equilibrium $\sigma \in \Sigma^*_\mu$, we also denote by $v_1(\sigma, Y)$ the equilibrium payoff obtained by party 1 when he starts by offering Y. We define upper bounds on party 1's equilibrium payoffs:

$$\bar{v}_1(\mu, Y) = \sup\{v_1(\sigma, Y), \sigma \in \Sigma^*_\mu\} \text{ and } \bar{v}_1(\mu) = \sup_Y \bar{v}_1(\mu, Y).$$

The lower bounds $\underline{v}_1(\mu, Y)$ and $\underline{v}_1(\mu)$ are defined similarly.

Finally, for any belief μ and equilibrium $\sigma \in \Sigma^*_\mu$, it will be convenient to denote by $\mu' = \mu'(\sigma; Y)$ party 1's equilibrium belief in the event where party 1 offers Y and party 2 rejects the offer Y.

We first state two preliminary results:

Lemma 2 *Assume that Proposition 10 holds for $\mu \geq \mu^n$ and that $\bar{v}_2 \leq v^k$, for some $k \leq n$. Then for any $\mu \in [\xi\mu^*, \mu^*)$ and $\sigma \in \Sigma^*_\mu$, and for any $l \leq k$,*
a. $Y > v^{l+1} \Rightarrow \mu'(\sigma, Y) \geq \mu^l$
b. $Y < v^l \Rightarrow \mu'(\sigma, Y) \leq \mu^l$

Lemma 2 thus describes how party 1's beliefs evolve in equilibrium, depending on the offer chosen by party 1. We also have:

Lemma 3 *Assume that Proposition 10 holds for $\mu \geq \mu^n$ and that $\bar{v}_2 \leq v^k$, for some $k \leq n$. Then for any $\mu \in [\xi\mu^*, \mu^*)$ and $\sigma \in \Sigma^*_\mu$, we have:*
a. $\underline{v}_1(\mu) \geq \max_{l \leq k+1} w^l(\mu)$

[24] This technique is similar to that used by Gul et al.(1986) and Fudenberg et al. (1985).

b. $\forall l \leq k$, $\forall Y \in (v^{l+1}, v^l)$, $v_1(\sigma, Y) < w^l(\mu)$
c.$\forall l \leq k+1$, *if* $Y = v^l$ *and* $\mu'(\sigma, Y) \geq \mu^*$, *then* $v_1(\sigma, Y) \leq w^l(\mu)$
d.$\forall l \leq k$, $\bar{v}_1(\mu, v^l) \leq w^l(\mu)$

Lemma 3 thus derives bounds on equilibrium payoffs. Before proving these two preliminary Lemmas, we show how they can be used to prove Proposition 10. We proceed in steps.

Step 1: $\bar{v}_2 \leq v^n$ and, for any $\mu \in [\xi\mu^*, \mu^*)$, $\underline{v}_1(\mu) \geq w(\mu)$.

Assume $\bar{v}_2 \leq v^k$, *for some* $k \leq n$. Observe that when $\mu < \mu^n$, then $w^0(\mu) < \cdots < w^l(\mu) < w^{l+1}(\mu) < \cdots < w^n(\mu)$. So an immediate implication of Lemma 3 is that if $k < n$, then $\underline{v}_1(\mu) \geq w^{k+1}(\mu) = \max_{l \leq k+1} w^l(\mu)$. Hence from b. and d., no offer $Y > v^{k+1}$ may be optimal for party 1, and party 2's equilibrium payoff is therefore at most equal to v^{k+1}, that is, $\bar{v}_2 \leq v^{k+1}$.

Starting from $\bar{v}_2 \leq v^0$, the above argument may be iterated until we obtain $\bar{v}_2 \leq v^n$, and Lemma 3 (part a.) then implies that for any $\mu \in [\xi\mu^*, \mu^*)$,

$$\underline{v}_1(\mu) \geq \max\{w^n(\mu), w^{n+1}(\mu)\} = w(\mu)$$

(where the equality holds because $\xi > 1 - \underline{\pi}$ implies $\mu > \mu^{n+2}$).[25]

Step 2: When $\mu \in [\xi\mu^*, \mu^*)$, only offers v^n and v^{n+1} may be optimal.

We use step 1 and Lemma 3 to conclude that

i) Only offers $Y = v^n$ and $Y \leq v^{n+1}$ may be optimal (by part b. and d.)
ii) If $Y = v^{n+1}$ and $\mu' \geq \mu^*$, then party 1 gets at most $w(\mu)$ in equilibrium. (by part c))
iii) If $Y < v^{n+1}$, then $\mu' < \mu^*$.[26]

Consider now the first date $\tilde{t}$ where party 2 accepts party 1's offer with probability at least equal to $1 - \mu^{\tilde{t}}/\mu^*$. At this date it must be that party 1 offers $Y^{\tilde{t}} \geq v^{n+1}$ (from iii.), hence computed from date $\tilde{t}$, party 1's continuation equilibrium payoff is at most equal to $w(\mu^{\tilde{t}})$ (from ii.). If $\tilde{t} \geq 2$, then party 1 gets at most

$$(1 - \frac{\mu}{\mu^{\tilde{t}}}) + \frac{\mu}{\mu^{\tilde{t}}}[\delta_1]^2 w(\mu^{\tilde{t}}),$$

which is smaller than $\xi + (1-\xi)[\delta_1]^2 w(\mu)$, hence strictly smaller than $w(\mu)$ when ξ is small enough. It follows that $\bar{v}_1(\mu) \leq w(\mu)$ and that v^n and v^{n+1} are the only two possible

[25] Because $\mu \geq \xi\mu^* \geq \xi\mu^{n+1} > (1 - \underline{\pi})\mu^{n+1} = \mu^{n+2}$.

[26] Indeed, if $Y < v^{n+1}$and $\mu' \geq \mu^*$, then the induction hypothesis implies that party 1 will offer at least v^n next time he makes an offer, so party 2 should rather reject Y (because $[\delta_2]^2 v^n = v^{n+1}$, hence $Y < [\delta_2]^2 v^n$), contradicting $\mu' > \mu$.

equilibrium offers made by party 1 when $\mu \in [\xi\mu^*, \mu^*)$. For a given $\mu \neq \mu^{n+1}$, which offer is optimal depends on how $w^{n+1}(\mu)$ compares to $w^n(\mu)$, that is, on how μ compares to μ^{n+1}. ■

Proof of Lemma 2: To prove **a.**, we first show that $\mu'(\sigma, Y) \geq \mu^*$. Indeed, if $\mu'(\sigma, Y) < \mu^*$, then when party 2 rejects Y, she obtains at most $[\delta_2]^2\bar{v}_2$ by definition of $\bar{v}_2$. Since $[\delta_2]^2\bar{v}_2 \leq [\delta_2]^2 v^k = v^{k+1} < Y$, party 2 when rational should accept Y with probability 1, contradicting $\mu'(\sigma, Y) < \mu^* < 1$.

Since $\mu'(\sigma, Y) \geq \mu^*$, the induction hypothesis applies to the continuation game. Assume that $\mu'(\sigma, Y) < \mu^l$. Then party 1's next equilibrium offer is at most equal to v^l, so party 2 obtains a payoff at most equal to $[\delta_2]^2 v^l = v^{l+1} < Y$ when she rejects Y; If rational she should strictly prefer to accept Y, contradicting $\mu'(\sigma, Y) < 1$.

To prove **b.**, assume that $\mu'(\sigma, Y) > \mu^l$. Then the induction hypothesis applies to the continuation game, and party 1's next equilibrium offer is at least equal to v^{l-1}. Party 2 may thus obtain a payoff at least equal to $[\delta_2]^2 v^{l-1} = v^l > Y$ by rejecting Y; if rational she should thus reject the offer Y, contradicting $\mu'(\sigma, Y) > \mu$. ■

Proof of Lemma 3: It will be convenient to define

$$w(\mu, Y, \mu') \equiv (1 - \mu/\mu')(1 - Y) + (\mu/\mu')[\delta_1]^2 w(\mu').$$

Lemma 2 implies that when party 1 offers $Y \in (v^{l+1}, v^l)$, then $\mu' = \mu^l$. Since $\mu^l \geq \mu^k \geq \mu^n \geq \mu^*$, the induction hypothesis applies to the continuation game, and party 1 therefore obtains a payoff equal to $w(\mu, Y, \mu^l)$. This payoff tends to $w^{l+1}(\mu)$ when Y tends to v^{l+1}, which proves a. Besides, for any $Y \in (v^{l+1}, v^l)$, $w(\mu, Y, \mu^l)$ is strictly smaller than $w^{l+1}(\mu)$ (which proves b.).

Lemma 2 also implies that when party 1 offers $Y = v^l$, then $\mu'(\sigma, Y) \in [\mu^l, \mu^{l-1}]$. The induction hypothesis applies (either because $\mu' \geq \mu^*$ is assumed, as in part c., or because $l \leq k \leq n$, as in part d.). In both cases, party 1 obtains a payoff equal to $w(\mu, v^l, \mu'(\sigma, Y))$, and it is easy to check[27] that

$$\max_{\mu' \in [\mu^l, \mu^{l-1}]} w(\mu, v^l, \mu') = w(\mu, v^l, \mu^{l-1}) = w^l(\mu), \tag{17}$$

which proves c. and d. ■

A Perfect Bayesian Nash equilibrium of the game without outside options: We now turn to the full description of an equilibrium of the game without outside option. Note that Proposition 10 already gives properties that any Perfect Bayesian Nash equilibrium

[27]Indeed, when $\mu' \in [\mu^l, \mu^{l-1}]$, $w(\mu') = w^{l-1}(\mu')$, and replacing $w(\mu')$ by $w^{l-1}(\mu')$ in the expression of $w(\mu, Y, \mu')$ permits to compute $\frac{\partial}{\partial \mu'} w(\mu, Y, \mu') = \frac{\mu}{\mu'^2}(v^{l-1} - v^l) > 0$.

must satisfy. In particular, if the current belief μ about the obstinacy of party 2 belongs to (μ^{n+1}, μ^n), then party 1's equilibrium offer must be equal to v^n. Our objective in what follows is to provide a *complete* description of a perfect Bayesian equilibrium.

Our candidate equilibrium strategy profile, denoted σ^*, is described by considering at any date, at any node and for any current belief $\mu > 0$ about the obstinacy of party 2 that party 1 holds *at that node*, the behavioral strategies of parties 1 and 2 (induced by the strategy profile).

Define N and the sequence $\{w^n, v^n, \pi^n, \mu^n, w^n(\mu)\}_{0 \leq n \leq N}$ as in page 25. We also let μ_0^* be the smallest belief μ for which $w^0(\mu) = 1 - \theta$.

It will be convenient to let Y^- denote the last offer made by party 1 (when party 2 is the proposer, it is the offer made in the previous period, and when party 1 is the proposer, it is the offer made two periods earlier). We will also denote by Y (resp. X) the offer made by party 1 (resp. 2) when he (respectively she) is the proposer.[28]

I. *At any (odd) date where party 1 moves first,*

a) Party 1's offer

- If $\mu \geq \mu^0$: party 1 offers θ if $\mu \geq \mu_0^*$, v^0 otherwise.

- If $\mu \in (\mu^{n+1}, \mu^n)$ with $n \geq 0$, then party 1 offers v^n.

- If $\mu = \mu^{n+1}$ and the last offer of party 1 (made two periods earlier) is Y^-, then party 1 offers v^{n+1} with probability $q(Y^-)$, and v^n with probability $1 - q(Y^-)$, where $q(Y^-)$ satisfies

$$Y^- = [\delta_2]^2 [q(Y^-) v^{n+1} + (1 - q(Y^-)) v^n]$$

and we choose $q(Y^-) = 1$ when there is no last offer (in period 1) or when $Y^- < v^{n+2}$.

d) Party 2's response

- If $\mu \geq \mu^0$: party 2 if rational accepts any offer $Y \geq v^0$.

- If $\mu \in [\mu^{n+1}, \mu^n)$, party 2 rejects any offer $Y < v^{n+1}$, and accepts with probability $1 - \mu/\mu^{k-1}$ any offer $Y \in [v^k, v^{k-1})$, when $0 \leq k \leq n+1$. (we set $\mu^{-1} = 1$ and $v^{-1} = \theta$).

II. At any (even) date where party 2 moves first, party 2 offers $X = 1 - \theta$ to party 1. After any offer $X \neq 1 - \theta$, the current belief becomes $\mu = 0$ and continuation play follows that described in Proposition 1. After the obstinate offer $X = 1 - \theta$, party 1 accepts the offer $(1 - \theta)$ if $\mu > \mu^0$ and rejects it if $\mu < \mu^0$. For $\mu = \mu^0$, and if the last offer made by party 1 (in the previous period) is Y^-, party 1 rejects the obstinate offer $(1 - \theta)$ with probability

[28] Except otherwise mentioned the probabilities of acceptance of party 2 are in expected terms. That is, they do cover the behavior of party 2 if obstinate. The behavioral acceptance probabilities of the rational party 2 are obtained by dividing these expected probabilities by $1 - \mu$ where μ is the current probability that 2 is obstinate.

$\widetilde{q}(Y^-)$ *where*

$$Y^- = \delta_2(1-\widetilde{q}(Y^-))\theta + [\delta_2]^2\widetilde{q}(Y^-)v^0.$$

and we choose $\widetilde{q}(Y^-) = 1$ *when there is no last offer (in period* 1*) or when* $Y^- < v^1$.

Under the above strategy profile, when $\mu \geq \mu^0$ is the current probability that party 2 is obstinate and it is party 1's turn to move first, party 1 obtains a payoff equal to

$$w^0(\mu) \equiv \max\{(1-\mu)(1-v^0) + \mu\delta_1(1-\theta), 1-\theta\}.$$

When initially the prior probability that party 2 is obstinate is given by $\varepsilon \in (\mu^{n+1}, \mu^n]$, with $n \geq 0$, party 1 starts by offering v^n, which is accepted with probability $1 - \frac{\varepsilon}{\mu^{n-1}}$. Then starts a phase where only offers made by party 1 are accepted with positive probability (party 2's offers are $(1-\theta, \theta)$ which are rejected by party 1). In this phase, the sequence of offers by party 1 and acceptance probabilities by party 2 is given by[29]

$$(v^{n-1}, \pi^{n-1}), ..., (v^0, \pi^0)$$

In the event where party 2 has rejected all these offers, party 1 believes that he faces the obstinate party with probability 1. Party 2 then offers $(1-\theta)$, which is accepted by party 1 with probability 1.

The expected payoff for party 1 associated with this sequence is precisely equal to

$$w^n(\varepsilon) = (1 - \frac{\varepsilon}{\mu^{n-1}})(1-v^n) + [\delta_1]^2 \frac{\varepsilon}{\mu^{n-1}} w^{n-1}. \tag{18}$$

Observe that party 1's offers v^n are such that party 2 is always indifferent between accepting the offer v^n today and accepting the offer v^{n-1} two periods later. Also, recall that the beliefs μ^k have been defined so that $w^n(\mu^{n+1}) = w^{n+1}(\mu^{n+1})$ thus ensuring that when $\mu = \mu^{n+1}$ party 1 is indifferent between offering v^n and v^{n+1}.

We have the following proposition:[30]

Proposition 11 *Let* ε *be the prior probability that party* 2 *is obstinate. Assume that there exists an integer* $n < N$ *such that* $\varepsilon \in (\mu^{n+1}, \mu^n]$. *Then the strategy profile* σ^* *as defined above is a Perfect Bayesian equilibrium of the game without outside options, and party* 1*'s equilibrium payoff is equal to* $w^n(\varepsilon)$.

[29] After each offer v^{k+1} (with $k \geq 1$) and rejection by party 2, the current belief becomes $\mu = \mu^k$, the last party 1's offer is $Y^- = v^{k+1}$, so $q(Y^-) = 1$ and party 1 next offers v^k with probability 1. After the offer v^1 and rejection by party 2, $\mu = \mu^0$ and $Y^- = v^1$, so $\widetilde{q}(Y^-) = 1$ and party 1 next offers v^0 with probability 1.

[30] Note that if $\theta > v_2^*$, then for any fixed ε and $\frac{1-\delta_2}{1-\delta_1}$, the condition of Proposition 11 holds for δ_2 close enough to 1. Indeed, as in the Proof of Proposition 2, choose n^* such that $(1-\underline{\pi})^{n^*} < \varepsilon$. We have $\mu^{n^*} \leq (1-\underline{\pi})^{n^*} < \varepsilon$, and since n^* is bounded above independently of δ_2, $[\delta_2]^{2(n^*+1)}\theta > v_2^*$ for δ_2 close enough to 1.

Proof. Under the condition of Proposition 11, $\varepsilon \in (\mu^{n+1}, \mu^n]$ and $\delta_2 v^n > v_2^*$. Under the strategy profile σ^*, party 1 starts with offer v^n and after any history, party 1 makes offers above v^n. It is therefore never optimal for party 2 to reveal herself at a date where she is the proposer (she would get only v_2^* by doing so). Since only sequences of offers $v^k, v^{k-1}, ..., v^0$, (with $k \leq n$) may arise in our candidate equilibrium, at any date where he responds to an offer v^k, the rational party 2 is always indifferent between accepting and rejecting the offer made by party 1. This is also true after an out of equilibrium offer $Y \geq v^{k+1}$ of party 1 (this follows from the definitions of $q(Y)$ and $\widetilde{q}(Y)$). After an out of equilibrium offer $Y < v^{k+1}$, party 2 obtains v^k two periods later (since then $q(Y) = 1$ -and $\widetilde{q}(Y) = 1$ in case $k = 0$), and it is therefore optimal for party 2 to reject the offer Y.

When party 1 makes an offer $Y \in [v^k, v^{k-1})$, with $k \geq 0$, the probability that party 2 accepts is $1 - \mu/\mu^{k-1}$. Party 1's continuation payoff (after the acceptance decision of party 2) is thus independent of the offer made. Hence only offers equal to θ or to v^k for some k may be optimal for party 1. The optimality of v^n when $\mu \in [\mu^{n+1}, \mu^n]$ follows from the properties of μ^n and the observation that for all k and $\mu < (>) \mu^{k+1}$, $w^{k+1}(\mu) > (<) w^k(\mu)$.

The belief μ_0^* is such that when it is her turn to move first party 1 is indifferent between offering θ (which is accepted with probability 1) and offering $\delta_2\theta$ (which is accepted by party 2 only when rational thus with probability $1 - \mu_0^*$) - see the above lemma.

The belief μ^0 is such that when it is party 2's turn to move first, party 1 is indifferent between accepting the offer $1 - \theta$ of party 2 and rejecting it (which results one period later in the payoff $w^0(\mu^0)$). This ensures the optimality of party 1's behavior when he responds to the offer $1 - \theta$ of party 2. ■

Appendix B (wars of attrition)

In this Appendix, we will prove Propositions 3 and 8, which both deal with the case of two-sided uncertainty. (Proposition 3 considers the case without outside options; Proposition 8 considers the case with one-sided outside option.)

In order to prove these Propositions, it is convenient to consider the following game $G_{\varepsilon_1, \varepsilon_2}$, which is analyzed in Proposition 9: Each party i may be obstinate (of type θ_i) with probability ε_i. Party 1 moves first. As long as each party $i = 1, 2$ behaves according to his obstinate type, the moves are the same as in the bargaining game without outside options as described in Section 2. When at date t party i has a behavior that differs from the obstinate type, the game stops; the payoffs to parties i and j are given by

$$(v_i^t, v_j^t) \in V_\mu^i \tag{19}$$

where μ denotes the current belief that party j is obstinate (both parties cannot reveal themselves as rational at the same time).

We will make the following assumptions on the sets V^i_μ, which will be satisfied in both the contexts of Propositions 3 and 8.

Assumption 1 *For $i = 1, 2$, there exist functions $w_1(\mu), w_2(\mu)$ and constants $\bar{w}_1, \bar{w}_2$ such that $\forall (v_i, v_j) \in V^i_\mu$:*

$$\mid v_i - w_i(\mu) \mid \leq B(1-\delta)$$
$$\mid v_j - \bar{w}_j \mid \leq B(1-\delta)$$

where B is independent of δ. Furthermore, there exists $\underline{w} > 0$ and $\Delta > 0$ such that for each i, the following properties hold:

$$w_i(\mu) \geq \underline{w} \ \forall \mu \in [0,1], \tag{20}$$

$$w_i'' \text{ exists and is bounded} \tag{21}$$

$$\bar{w}_i - w_i(\mu) + \mu w_i'(\mu) \geq \Delta \ \forall \mu \in [0,1]. \tag{22}$$

For Proposition 3, the analysis of the ensuing equilibrium outcome after one party has revealed himself corresponds to the one-sided uncertainty case and is analyzed in Proposition 2. As Appendix A shows, Assumption 1 is satisfied with $w_i(\mu) = 1 - \theta_j$, and $\bar{w}_j = \theta_j$ for $i = 1, 2$ and $j \neq i$.

For Proposition 8, the analysis of the ensuing equilibrium outcome after one party has revealed himself corresponds to the one-sided uncertainty case and has been described in the main text. Assumption 1 is satisfied with $w_i(\mu) = (1-\mu)v^* + \mu\delta v_i^{out}$ and $w_j(\mu) = 1 - \theta$; $\bar{w}_i = \theta$ and $\bar{w}_j = \delta v^*$.

The next proposition analyzes the equilibrium values of the game $G_{\varepsilon_1,\varepsilon_2}$ under Assumption 1. To present the proposition, it will be convenient to let:

$$a_i(\mu) = \frac{(1-\delta_i)w_i(\mu)}{\bar{w}_i - w_i(\mu) + \mu w_i'(\mu)}$$

and

$$\phi_i(\mu) = \int_\mu^1 \frac{d\mu'}{\mu' a_i(\mu')},$$

which are well defined functions thanks to Assumption 1. We have the following Proposition.

Proposition 12 *Let $\frac{1-\delta_2}{1-\delta_1}$ be fixed and equal to r, and let $\delta = \min(\delta_1, \delta_2)$. Consider any game $G_{\varepsilon_1,\varepsilon_2}$ as described above such that all sets V^i_μ for $i = 1, 2$ satisfy Assumption 1. If*

$\phi_i(\varepsilon_j) > \phi_j(\varepsilon_i)$, choose z such that $\phi_i(z) = \phi_j(\varepsilon_i)$,[31] *and $\pi^* = 1 - \varepsilon_j/z$. In any Perfect Bayesian Equilibrium of $G_{\varepsilon_1,\varepsilon_2}$, the payoffs (v_i, v_j) of parties i and j (when rational) satisfy:*

$$\mid v_i - [(1-\pi^*)w_i(z) + \pi^*\bar{w}_i] \mid < b(1-\delta)^{1/2} \text{ and } \mid v_j - w_j(\varepsilon_i) \mid < b(1-\delta)^{1/2}$$

for some constant b independent of δ and of the equilibrium considered.

Before proving this key Proposition, we check that Propositions 3 and 8 are immediate Corollaries of Proposition 12.

Proof of Proposition 3: Under the conditions of Proposition 3, we have that $a_i(\mu) = a_i = \frac{(1-\delta_i)(1-\theta_j)}{\theta_i-(1-\theta_j)}$ and $\phi_i(\mu) = -\frac{1}{a_i}\ln\mu$ as indicated in the main text. ■

Proof of Proposition 8: Under the conditions of Proposition 8, $\phi_j(\mu) = -\frac{\theta-v^*}{1-\theta}\frac{1}{1-\delta}\ln\mu$ (remember that $1 - \delta v^* = v^*$). We also have

$$a_i(\mu) = \frac{(1-\delta)[(1-\mu)v^* + \mu\delta v_i^{out}]}{\theta - v^*} \geq \frac{(1-\delta)\delta v_i^{out}}{\theta - v^*}$$

for all $\mu \in (0, 1)$. Thus we have:

$$\phi_i(\mu) \leq -\frac{\theta - v^*}{\delta v_i^{out}}\frac{1}{1-\delta}\ln\mu = d\phi_j(\mu) \text{ with } d = \frac{1-\theta}{\delta v_i^{out}} < 1.$$

It follows that $\phi_j(\varepsilon) > \phi_i(\varepsilon)$ and the belief z that solves $\phi_i(\varepsilon) = \phi_j(z)$ satisfies $z \geq \varepsilon^d$. Hence $\pi^* = 1 - \varepsilon/z = 1 - \varepsilon^{1-d}$ and Proposition 12 then implies that parties equilibrium payoffs are equal to $v^* + O(\varepsilon^c) + O((1-\delta)^{1/2})$, with $c = 1 - d > 0$, as desired. ■

Proof of Proposition 12: We let N^t denote the event where neither party reveals himself as rational before t, we let $R_k^{t,m}$ denote the event where party k reveals himself as rational during $\{t, ..., t+m-1\}$, and we let ξ_i denote the strategy for party i that consists in mimicking the obstinate type.

Consider a Perfect Bayesian equilibrium σ of $G_{\varepsilon_1,\varepsilon_2}$. We define the probability

$$P_j^{t,m} \equiv \Pr\{R_j^{t,m} \mid N^t, \xi_i, \sigma_j\}$$

that party j reveals himself as rational during $\{t, ..., t+m-1\}$ given that neither party has revealed himself before t and that party i mimics the obstinate type. It will be convenient to let

$$\mu_k^t = \Pr\{k \text{ is obstinate} \mid N^t, \sigma\}$$

[31] The number z is uniquely defined because ϕ_i is a decreasing function of μ (the function a_i is positive by (20) and (??)).

denote the probability that k is obstinate at date t given that neither party has revealed himself as rational before t. Bayes' law implies:

$$P_j^{t,m} = 1 - \mu_j^t / \mu_j^{t+m}$$

We also let $u_i^{t,m} \equiv \sum_{s=0}^{m-1} \delta^s u_i^{t+s}$, where u_i^{t+s} is the payoff obtained by party i at date $t+s$. We define

$$\begin{aligned} v_i^t &\equiv E[u_i^{t,\infty} \mid N^t, \sigma] \text{ and} \\ \bar{v}_i^{t,m} &\equiv E[u_i^{t,m} \mid N^t, R_j^{t,m}, \xi_i, \sigma_j]. \end{aligned}$$

The value v_i^t is thus Party i's *continuation equilibrium payoff* under the event N_t, and $\bar{v}_i^{t,m}$ is the expected payoff obtained (under the event N_t) by party i in equilibrium during the next m periods when he mimics the obstinate type. If $\mu_i^{t+m} < 1$, then it is optimal for party i to mimic the obstinate type until date $t+m$ at least. Therefore we have:

$$v_i^t = P_j^{t,m} \bar{v}_i^{t,m} + (1 - P_j^{t,m}) \delta_i^m v_i^{t+m} \tag{23}$$

It will also be convenient to let $T_i \equiv \sup\{t, \mu_i^t < 1\}$ denote the last (possibly infinite) date at which party i reveals himself as rational and $T \equiv \max\{T_1, T_2\}$. Note that if T_i is finite, then at $T_i + 1$, party i may only be obstinate, so it is optimal for the (still) rational party j to concede to party i (because this is the best response against an opponent known to be obstinate with probability 1), so that $T_j \leq T_i + 1$, which further implies $T \geq T_k \geq T - 1$ for $k = 1, 2$.

The rest of this proof will make extensive use of equation (23). To see informally the implication of equation (23), let us consider the case where v_i^t and $\bar{v}_i^{t,m}$ are constant over time and respectively equal to w_i and $\bar{w}_i$. Then equation (23) implies that

$$\frac{1}{m} P_j^{t,m} \approx (1 - \delta_i) \frac{w_i}{\bar{w}_i - w_i}.$$

Since Bayes Law implies $\frac{\mu_j^t}{\mu_j^{t+m}} = 1 - P_j^{t,m}$, or equivalently $-\operatorname{Log} \mu_j^{t+m} + \operatorname{Log} \mu_j^t = \operatorname{Log}(1 - P_j^{t,m})$, and since $\operatorname{Log}(1 - P_j^{t,m}) \approx -P_j^{t,m}$, we conclude that

$$\mid \operatorname{Log} \mu_j^t \mid \approx (1 - \delta_i)(T - t) \frac{w_i}{\bar{w}_i - w_i}, \tag{24}$$

which relates the time it takes for party j to build a reputation for obstinacy (the time after which $\mu_j^t = 1$) to the parameters of the model.

The next Lemma shows that in equilibrium v_i^t and $\bar{v}_i^{t,m}$ remain close to respectively $w_i(\mu_j^t)$ and $\bar{w}_i$ (instead of being exactly equal to w_i and $\bar{w}_i$), and then generalizes the relationship (24) to this more general case.

Lemma 4 *Let $m^* = int(1-\delta)^{-1/2}$.*[32] *There exist constants δ_0, m_0, $P_0, \bar{B}, c, C$ such that for any discount $\delta \geq \delta_0$, and any perfect Bayesian equilibrium of the game, the following properties hold:*
a) $\forall t < T-1$, $P_1^{t,m_0} > 0$ and $P_2^{t,m_0} > 0$.
b) $\forall t \geq m_0$, $P_i^{t,m} < P_0(1-\delta)$, $i \in \{1,2\}$.
c) $\forall t \geq m_0$, $\mid v_i^t - w(\mu_j^t) \mid \leq \bar{B}P_0(1-\delta)$.
d)$\forall t \in \{m^*, ..., T-m^*\}$, $\mid P_j^{t,m^*} - m^* a_i(\mu_j^t) \mid \leq c(1-\delta)^{1/2}$
e) $\mid T - \phi_i(\mu_j^{m^*}) \mid \leq CT(1-\delta)^{1/2}$

We may now conclude the proof of Proposition 12. We distinguish two cases, depending on which party is the first one to reveal himself as rational with positive probability in equilibrium. First assume that party j is the first party to reveal himself as rational with positive probability in equilibrium. By assumption 1, party j's equilibrium payoff must be at most equal to $w_j(\varepsilon_i) + B(1-\delta)$. Now by equation (23) and Lemma 4 (step c), party j's equilibrium payoff must also satisfy:

$$v_j = P_i^{1,m^*}\bar{w}_j + (1 - P_i^{1,m^*})w_j(\mu_i^{m^*}) + O((1-\delta)^{1/2}). \tag{25}$$

Hence P_i^{1,m^*} must be comparable to $(1-\delta)^{1/2}$.[33]

Now observe that

$$\mid \phi_i^{-1}[\phi_j(\mu_i^{m^*})] - \phi_i^{-1}[\phi_j(\varepsilon_i)] \mid \leq \frac{\sup \phi_j'}{\inf \phi_i'} \mid \mu_i^{m^*} - \varepsilon_i \mid \tag{26}$$

and that

$$\mid \phi_i^{-1}[\phi_j(\mu_i^{m^*})] - \phi_i^{-1}[\phi_i(\mu_j^{m^*})] \mid \leq \frac{1}{\inf \phi_i'} \mid \phi_j(\mu_i^{m^*}) - \phi_i(\mu_j^{m^*}) \mid \tag{27}$$

Since ϕ_k' is comparable to $1/(1-\delta_k)$ and since P_i^{1,m^*} is comparable to $(1-\delta)^{1/2}$, the right hand side of (26) is comparable to $(1-\delta)^{1/2}$, and by Lemma 4 step e, we also conclude that right hand side of (27) is comparable to $(1-\delta)^{1/2}$. Combining inequalities (26) and (27) therefore implies that $\mu_j^{m^*}$ is approximately equal to $\phi_i^{-1}[\phi_j(\varepsilon_i)]$ (up to a term comparable to $(1-\delta)^{1/2}$), which further implies the desired result that P_i^{1,m^*} is approximately equal to π^*.

Now consider the other case where party i is the first party to reveal himself with positive probability in equilibrium. Then the above analysis implies that $\mu_j^{m^*}$ is approximately equal to $\phi_i^{-1}[\phi_j(\varepsilon_i)]$. Since $\phi_i^{-1}[\phi_j(\varepsilon_i)] < \varepsilon_j$ (by assumption) and since $\mu_j^{m^*} \geq \varepsilon_j$, $\mu_j^{m^*}$ must also be approximately equal to ε_j. And inequalities (26) and (27) then imply again that P_i^{1,m^*} is approximately equal to π^* (up to a term comparable to $(1-\delta)^{1/2}$). ■

[32] $int(x)$ denotes the largest integer no greater than x.

[33] Indeed, $\mu_i^{m^*} = \frac{\varepsilon_i}{1-P_i^{1,m^*}}$, and from Assumption 1, the derivative of the right-hand side of (25) wrt P_i^{1,m^*} is bounded from below by some $\Delta > 0$. Thus $v_j \geq w_j(\varepsilon_i) + \Delta P_i^{1,m^*} + O((1-\delta)^{1/2})$ implying that $P_i^{1,m^*} = O((1-\delta)^{1/2})$.

Proof of Lemma 4: a) We first prove that there exists m_1 that can be set independently of δ (above a threshold δ_0) such that $\forall t \leq T - m_1$, either $P_1^{t,m_1} > 0$ or $P_2^{t,m_1} > 0$. Choose t and $m \geq 1$ such that $t \leq T - m$. Assume that $P_k^{t,m} = 0$ for $k \in \{1, 2\}$, and let i be the first party to reveal himself as rational with positive probability after date $t + m - 1$, say at date $t + m + m'$ with $m' \geq 0$ (such a date exists by definition of T and because $t + m \leq T$). By Assumption 1, we must have $v_i^{t+m} \leq \delta^{m'}[w_i(\mu_j^{t+m+m'}) + B(1-\delta)]$. Party i may also secure a payoff at least equal to $w_i(\mu_j^t) - B(1-\delta)$ by revealing himself as rational at t. By assumption, $P_j^{t,m+m'} = 0$, hence $\mu_j^{t+m+m'} = \mu_j^t$, and equation (23) implies:

$$w_i(\mu_j^t) - B(1-\delta) \leq \delta^m[w_i(\mu_j^t) + B(1-\delta)],$$

which gives a contradiction for $m \geq m_1$ and $\delta \geq \delta_0$ when $m_1 = int(4B/\underline{w}) + 1$, (recall that $\underline{w} \leq \min_{i,\mu} w_i(\mu)$), and when δ_0 satisfies $[\delta_0]^{m_1} > 1/2$.

We now show that property a) holds with $m_0 = 2m_1$. Clearly, that property holds for $t > T - 2m_1$, with $t < T - 1$, by definition of T and because $T - 1 \leq T_i \leq T$. Consider now $t \leq T - 2m_1$. We know from above that $P_i^{t+m_1,m_1} > 0$ for some i. If $P_j^{t,2m_1} = 0$, then the above argument applies again: when at date t party i chooses to mimic the obstinate type until date $t + m_1$ at least, he gets a payoff at most equal to $\delta^{m_1}[w_i(\mu_j^t) + B(1-\delta)]$, which is strictly smaller than what he would get by revealing himself as rational at t. Contradiction (because $t < T$) .

b) Consider a date $t' \in \{t - m_0, t - 1\}$ where party i reveals himself as rational first with positive probability (such a date exists from a). Assumption 1 implies:

$$v_i^{t'} \leq w_i(\mu_j^{t'}) + B(1-\delta) \tag{28}$$

Now let $P \equiv P_j^{t',m_0+t-t'}$. When j reveals himself first as rational, party i obtains a payoff at least equal to $\bar{w}_i - B(1-\delta)$ by assumption 1. So equation (23) implies:

$$v_i^{t'} \geq \delta^{m_0+t-t'}[P(\bar{w}_i - B(1-\delta)) + (1-P)v_i^{t+m_0}]$$

Since $t - t' \leq m_0$, since $v_i^{t+m_0} \geq w(\mu_j^{t+m_0}) - B(1-\delta)$ and since $\mu_j^{t+m_0} = \mu_j^{t'}/(1-P)$ by Bayes' law, we have:

$$v_i^{t'} \geq \delta^{2m_0}[P\bar{w}_i + (1-P)(w_i(\frac{\mu_j^{t'}}{1-P})) - B(1-\delta))].$$

The derivative with respect to P of the bracketed term is equal to $\bar{w}_i - w_i(\mu') + \mu' w_i'(\mu')$ with $\mu' = \frac{\mu_j^{t'}}{1-P}$, which is bounded from below by Δ by Assumption 1. Since the value of the bracketed term at $P = 0$ is $w_i(\mu_j^{t'}) - B(1-\delta)$, we have:

$$v_i^{t'} \geq \delta^{2m_0}[w_i(\mu_j^{t'}) + \Delta P - B(1-\delta)] \tag{29}$$

Combining (28) and (29) yields the desired upperbound on P, hence on P^{t+m_0}.[34]

c) Consider a date $t < T - m^*$, where $m^* = int(1-\delta)^{-1/2}$. Party i reveals herself with positive probability at some date $t' \in \{t + m^*, t + m^* + m_0\}$ (from a), hence from equation (23), we get

$$v_i^t \leq P_j^{t,2m_0} + (1 - P_j^{t,2m_0}) w_i(\frac{\mu_j^t}{1 - P_j^{t,2m_0}})$$

Let $\bar{B}$ be an upperbound on the derivative of the right hand side with respect to $P_j^{t,2m_0}$. Since $P_j^{t,2m_0} \leq P_0(1-\delta)$, we obtain $v_i^t \leq w_i(\mu_j^t) + \bar{B}P_0(1-\delta)$ as desired.

d) Let us rewrite equation (23) as

$$v_i^t - \delta_i^{m^*} v_i^{t+m^*} - P_j^{t,m^*}[\bar{v}_i^{t,m} - \delta_i^{m^*} v_i^{t+m^*}] = 0 \tag{30}$$

The value $v_i^{t+m^*}$ is close to $w(\mu_j^{t+m^*})$ by Assumption 1, and (noting that $\mu_j^{t+m^*} = \frac{\mu_j^t}{1-P_j^{t,m^*}}$ by Bayes' law), a Taylor expansion of $w_i(\frac{\mu_j^t}{1-P_j^{t,m^*}})$ yields

$$| v_i^{t+m^*} - w_i(\mu_j^t) + \mu_j^t w_i'(\mu_j^t) | \leq c_0 [P_j^{t,m^*}]^2$$

for some constant c_0 (independent of δ and of the equilibrium considered).[35] Since party i obtains a payoff close $\bar{w}_i$ at any date where party j reveals herself first (by Assumption 1), we have $| \bar{v}_i^{t,m} - \bar{w}_i | \leq c_1(1 - [\delta_i]^{m^*})$ for some constant c_1, hence (30) implies:

$$| m^*(1-\delta_i) w_i(\mu_j^t) - P_j^{t,m^*}[\bar{w}_i - w_i(\mu_j^t) + \mu_j^t w_i'(\mu_j^t)] | \leq c_2 \max\{[P_j^{t,m^*}]^2, [m^*(1-\delta_i)]^2\}$$

for some constant c_2. Given that $P_j^{t,m^*} \leq P_0 \frac{m^*}{2m_0}(1-\delta)$ and that $m^* = int(1-\delta)^{-1/2}$, we obtain the desired inequality.

e) We let $\Lambda = \sup | \mu_j^{t+m^*} - \mu_j^t |$. It follows from Bayes' law and step b. that $\Lambda \leq c_3 m^*(1-\delta)$. Using step d) and Bayes' law to replace P_j^{t,m^*} by $1 - \mu_j^t/\mu_j^{t+m^*}$, we get

$$| m^* - \frac{\mu_j^{t+m^*} - \mu_j^t}{\mu_j^t a_i(\mu_j^t)} | \leq c_4$$

For any function $\eta(\mu)$ with bounded derivatives,

$$| \int_{\mu^t}^{\mu^{t+m^*}} (\eta(\mu) - \eta(\mu^t)) d\mu | \leq \Lambda^2 \sup | \eta'(\mu) |, \tag{31}$$

[34] For $\delta \geq \delta_0$, we get $P \leq \frac{2(m_0+B)}{\delta^{2m_0}}(1-\delta) \leq 8(m_0+B)(1-\delta)$.

[35] The constant c_0 as well as all other constants $c_1, ..., c_6$ and C to be used in the rest of this proof are independent of δ and of the equilibrium considered.

hence, applying (31) to $\eta(\mu) = \frac{1}{\mu a_i(\mu)}$ (this function has derivatives bounded by $\frac{c_5}{1-\delta_i}$ on $[\varepsilon_j, 1]$ thanks to Assumption 1), it follows that

$$| m^* - \int_{\mu^t}^{\mu^{t+m^*}} \frac{d\mu}{\mu a_i(\mu)} | \leq c_4 + c_5 \frac{\Lambda^2}{1-\delta_i} \leq c_6$$

The above inequality holds for any date $t_s = t + sm^* < T$. Choosing s^* such that $t_{s^*+1} > T \geq t_{s^*}$, we get (since $s^* \sim T/m^*$):

$$| T - t - \phi_i(\mu_j^t) | \leq m^* + s^* c_6 \leq CT(1-\delta)^{1/2}. \blacksquare$$

Appendix C (delayed outside options)

This Appendix is devoted to the analysis of the game where only one party (party 2) may be obstinate and party 1 has an outside option that becomes available at date T. After an informal description of the structure of equilibrium behavior, we present properties shared by any perfect Bayesian Nash equilibrium of the game. Proposition 6 will be derived as a corollary of these properties. For completeness we will also carefully describe a Perfect Bayesian Nash equilibrium of this game.

We have seen (in Proposition 5) that when the outside option is not delayed (and satisfies $v_1^{out} > 1 - \theta$), party 1 offers δv^* to party 2 in equilibrium. We have also seen that in the absence of outside options, party 1 makes an offer v^n for some n (close to θ when δ is close to 1) to party 2.

When the outside option is delayed and $\delta^T v_1^{out} > 1 - \theta$, it will be easy to check that in any perfect Bayesian equilibrium, party 1 behaves as if the outside option was immediately available and offers δv^* to party 2.

When the delay is large (so that $\delta^T v_1^{out} < 1 - \theta$), waiting for the outside option does not dominate accepting party 2's obstinate demand, so the analysis of Proposition 5 does not apply. Nevertheless if the probability that party 2 is obstinate is very small , it should take party 2 many periods before building a reputation for obstinacy, and party 1's option to wait for the outside option should thus become credible.

We will show that in any perfect Bayesian equilibrium, at any date t, party 1's equilibrium offer depends on his current belief about party 2's obstinacy in a very simple way. Below some threshold $\widetilde{\mu}_t$, party 1 behaves as if the outside option was immediately available and offers δv^* to party 2. Above that same threshold, party 1 behaves as if the outside option was not available, and offers v^n (for some n) to party 2. And at the threshold $\widetilde{\mu}_t$, party 1 is just indifferent between offering δv^* and v^n.

Let us make two additional statements concerning equilibrium behavior, which we will prove in this Appendix.

1) The thresholds $\widetilde{\mu}_t$ increase with t.[36]

2) As long as party 1 offers δv^* and party 2 rejects party 1's offers, the sequence of equilibrium beliefs held by party 1 at dates where he makes an offer coincides with $\widetilde{\mu}_3$, $\widetilde{\mu}_5, ..., \widetilde{\mu}_t, ...$[37].

Our main objective in this Appendix will be to prove that all perfect Bayesian equilibrium have the structure outlined above, and to compute a lower bound on $\widetilde{\mu}_3$, which will in turn imply that party 2 either accepts δv^* at date 1 or reveals himself at date 2 with probability equal to $1 - \varepsilon/\widetilde{\mu}_3$ (which is close to 1 when ε is close to 0).

We start with some notation. The integer N, the sequence $\{w^n, \mu^n, v^n, \pi^n, w^n(\mu)\}_{0 \leq n \leq N}$ and the function $w(\cdot)$ are defined as in Appendix A.

Let τ be the *first date where party* 1 *makes an offer and*

$$\delta^{T-\tau-1} v_1^{out} > 1 - \theta. \tag{32}$$

Date $\tau + 1$ is thus the first date where party 2 makes an offer and for which party 1 strictly prefers to wait for the outside option rather than accepting party 2's inflexible demand. For any $t \geq \tau$, we define

$$w_t(\mu) = \max\{(1-\mu)v^* + \mu\delta^{T-t}v_1^{out}, 1 - \theta\}.$$

That is, $w_t(\mu)$ corresponds to the expected gain of party 1 when i) he offers δv^* to party 2, ii) party 2 if rational accepts, and iii) he waits for the outside option in case party 2 rejects.

Next, for any date $t \leq \tau - 2$ where party 1 makes an offer, we define by induction on t:

$$\mu_{t+2}^* = \sup\{\mu \leq 1, \delta w_{t+2}(\mu) > \max\{1-\theta, \delta w(\mu)\}\} \tag{33}$$

$$w_t(\mu) = (1 - \frac{\mu}{\mu_{t+2}^*})v^* + \frac{\mu}{\mu_{t+2}^*}\delta^2 w_{t+2}(\mu_{t+2}^*) \tag{34}$$

[36] The reason is that as t increases, there is less time for party 2 to build a reputation for obstinacy.

[37] The reason is as follows: if at date t, $\mu_t < \widetilde{\mu}_t$, party 1 is supposed to offer δv^*. Rather than accepting such an offer, party 2 should reveal herself at date $t-1$. (hence we should have $\mu_t = 1$). And if $\mu_t > \widetilde{\mu}_t$, party 1 is supposed to offer v^n (close to θ), so rather than accepting δv^* at some earlier date, party 2 should wait for date t before accepting any offer.

We also let $\widetilde{\mu}_t$ denote the belief that solves[38]

$$\widetilde{\mu}_t = \sup\{\mu \leq 1, w_t(\mu) > w(\mu)\}.$$

As mentioned before, in equilibrium, it will turn out that when $\mu \leq \widetilde{\mu}_t$, it is optimal for party 1 to offer δv^* to party 2. The probability $(1 - \frac{\mu}{\mu^*_{t+2}})$ will correspond to the probability that party 2 accepts this offer, and μ^*_{t+2} thus corresponds to the belief held by party 1 in the event where party 2 rejects the offer; and the value $\delta w_{t+2}(\mu^*_{t+2})(= \max\{1-\theta, \delta w(\mu^*_{t+2})\})$ may thus be interpreted as party 1's continuation payoff after the offer δv^* has been rejected, in the subgame where party 2 makes an offer.

We have the following Proposition, where for convenience we set $\mu^*_t = \widetilde{\mu}_t = 1$ for all $t > \tau$.

Proposition 13 *Define $\mu^*_t, \widetilde{\mu}_t$ and $w_t(\mu)$ as above. Consider any date $t < T$ where party 1 makes an offer and let μ be party 1's belief about party 2's obstinacy. party 1's equilibrium payoff is uniquely defined and equal to $v^t_1(\mu) = \max\{w_t(\mu), w(\mu)\}$. Besides, in equilibrium, a) if $\mu > \widetilde{\mu}_t$, the continuation equilibrium is an equilibrium of the game without outside options, b) if $\mu < \widetilde{\mu}_t$, party 1 offers δv^* to party 2, who accepts with probability $1 - \mu/\mu^*_{t+2}$; c) Party 2's equilibrium payoff is at most equal to the payoff she obtains in the game without outside options.*

Before proving Proposition 13, we derive a lower bound on the thresholds μ^*_t, and complete the proof of Proposition 6.

Lemma 5 *Choose μ^* so that $\ln \mu^* = \alpha \ln \frac{v_1}{1-\theta}$, where $\alpha = \frac{1-\theta}{1/2-(1-\theta)}$. There exists a constant c independent of δ, τ such that for all $t \leq \tau$, $\mu^*_t \geq \mu^* - c(1-\delta)$.*

Note that since $\widetilde{\mu}_t \geq \mu^*_t$, Lemma 5 gives us a lower bound on the threshold $\widetilde{\mu}_t$ as well.

Proof. The sequence μ^*_t satisfies:

$$\mu^*_t/\mu^*_{t+2} \geq \frac{v^* - w(\mu^*_{t+2})}{v^* - \delta^2 w(\mu^*_t)}$$

Taking the product of these expressions, we get:

$$\mu^*_t \geq \mu^*_\tau \frac{v^* - w(\mu^*_\tau)}{v^* - \delta^2 w(\mu^*_t)} \prod_{s, t < t+2s < \tau} \frac{v^* - w(\mu^*_{t+2s})}{v^* - \delta^2 w(\mu^*_{t+2s})}$$

[38]These threshold beliefs μ^*_t and $\widetilde{\mu}_t$ are well defined and if $\widetilde{\mu}_{t+2} \in [\mu^n, \mu^{n-1})$, $n \geq 1$, then they satisfy:

$$\mu^{n+1} < \mu^*_t \leq \widetilde{\mu}_t < \mu^*_{t+2}.$$

Indeed, it is easy to check that if $\widetilde{\mu}_{t+2} \in [\mu^n, \mu^{n-1})$, $n \geq 1$, then $w_t(\mu^{n+1}) > w(\mu^{n+1})$. It is also easy to check that in the interval $[\mu^{n+1}, \mu^{n-1}]$, $\frac{\partial}{\partial \mu} w_t(\mu) - \frac{\partial}{\partial \mu} w(\mu) = \frac{\delta v^* - v^n}{\mu^*_{t+2}} + O(1-\delta)$, hence the equation $w_t(\mu) = w(\mu)$ has a unique solution in this interval.

For any $\mu > 0$, $w(\mu) \geq \underline{w} \equiv 1-\theta$, and for any $\mu \geq \frac{\mu^*}{2}$, there exists a constant κ independent of δ and τ such that $w(\mu) \leq \bar{w} \equiv 1-\theta+\kappa(1-\delta)$. Assume that $\mu^*_{t+2} > \frac{\mu^*}{2}$. Then μ^*_t satisfies

$$\ln \mu^*_t \geq \frac{\tau - t}{2} \ln[1-\alpha(1-\delta^2)] - d[(1-\delta) + \tau(1-\delta)^2] \tag{35}$$

for some constant d independent of δ and τ. Besides by definition of τ, $\delta^{T-\tau} v_1^{out} \geq 1-\theta$. Since $\delta^T v_1^{out} = \underline{v}_1$, we thus have

$$\tau \geq \frac{\ln \frac{\underline{v}_1}{1-\theta}}{\ln \delta}.$$

Replacing τ by this lower bound in (35), and recalling that $\ln \mu^* = \alpha \ln \frac{\underline{v}_1}{1-\theta}$, we obtain $\mu^*_t \geq \mu^* - c(1-\delta)$ for some constant c independent of δ and τ. ■

We are now ready to complete the proof of Proposition 6.

Proof of Proposition 6: Under the condition of Proposition 6, Proposition 13 implies that party 1 offers δv^* to party 2, who accepts with probability $1-\varepsilon/\mu^*_3 = 1-\varepsilon/\mu^* + O(1-\delta)$. ■

Proof of Proposition 13: We first consider dates larger than τ. We will then proceed backward from date τ.

Dates $t \geq \tau$: At any date $t \geq T$ where party 1 makes an offer, Proposition 4 applies, and party 2's equilibrium payoff is at most equal to δv^*, whatever the current belief μ_t is. This bound on party 2's equilibrium payoff is central to our argument. We have the following Lemma:

Lemma 6 *Consider a date $t \geq \tau$ where party 1 makes an offer. Assume that for any belief $\mu_t < 1$, party 2's equilibrium payoff is at most equal to δv^*. Then at date $t-2$, party 2 if rational behaves as follows: a) he accepts any offer $Y_{t-2} > \delta v^*$; and b) he rejects any offer $Y_{t-2} < \delta v^*$ and reveals himself at date $t-1$.*

Lemma 6 implies (under the stated conditions) that at date $t-2$, if the current belief of party 1 is μ, party 1 may secure

$$w_{t-2}(\mu) = (1-\mu)v^* + \mu\delta^{T-t+1} v_1^{out}$$

by making an offer $Y_{t-2} > \delta v^*$ arbitrarily close to δv^*. Lemma 6 also implies that if $\mu < 1$, the only offers that may be optimal are δv^* and θ. If $t-2 > \tau$, then by definition of τ, $w_{t-2}(\mu)$ is strictly larger than $1-\theta$, implying that in equilibrium, if $\mu < 1$, party 1 offers δv^* to party 2 who accepts it if rational. Therefore at date $t-2$, for any belief $\mu_{t-2} < 1$, party 2's equilibrium payoff is at most equal to δv^*.

Lemma 6 may thus be applied again, until date τ is reached. At date τ, the threshold $\widetilde{\mu}_\tau$ is strictly smaller than 1 (since $w_\tau(1) < 1-\theta$). And party 1 either offers δv^* or θ depending on whether μ is smaller or larger than $\widetilde{\mu}_\tau$.

Dates $t < \tau$. In what follows, we assume that Proposition 13 holds at date t onwards, and we show that it also holds at $t-2$. We let μ_{t-2} denote the belief at date $t-2$. We consider the subgame when party 1 has offered Y_{t-2} at $t-2$, and let μ_{t-1} and μ_t denote the belief that obtain in equilibrium respectively in the event where party 2 rejects Y_{t-2} and in the event where party 2 rejects Y_{t-2} and offers the obstinate demand (at $t-1$). The following Lemma is key to our induction argument. It shows how party 1's belief evolves when Y_{t-2} is offered and party 2 mimics the obstinate type.

Lemma 7 *Consider $t < \tau$ and assume that Proposition 13 holds from date t on. We let n be such that $\widetilde{\mu}_t \in [\mu^n, \mu^{n-1})$, with $\mu^{-1} = 1$, and we assume that $\mu_{t-2} \leq \mu_t^*$. Then we must have:*

a. *If $Y_{t-2} < \delta v^*$, then $\mu_{t-1} = \mu_{t-2}$ and $\mu_t = \mu_t^*$.*

b. *If $Y_{t-2} > \delta v^*$, $\mu_{t-1} = \mu_t$*

c. *If $Y_{t-2} < v^n$, $\mu_t = \mu_t^*$*

d. *If $Y_{t-2} > v^n$, $\mu_{t-1} = \mu_t > \mu_t^*$.*

Assuming that Proposition 13 holds from date t on (induction hypothesis), Lemma 7 allows us to derive bounds on parties' equilibrium payoffs, and to derive party 1's equilibrium offer as a function of his current belief.

1. *Party 1's equilibrium payoff is at least equal to* $\max\{w_{t-2}(\mu), w(\mu)\}$.

From b. and c., party 1 may secure $w_{t-2}(\mu)$ by making an offer arbitrarily close to δv_+^*. When $\mu \leq \widetilde{\mu}_{t-2}$, $w_{t-2}(\mu)$ is larger than $w(\mu)$. So to conclude this step, we only need to check that party 1 may secure $w(\mu)$ when $\mu > \widetilde{\mu}_{t-2}$. From d. and the induction hypothesis, offers $Y_{t-2} > v^n$ lead to continuation equilibria that are equilibria of the game without outside options. From Proposition 10, and for any $k \leq n$, the only continuation belief consistent with the offer v_+^k is $\mu_{t-1} = \mu^{k-1}$, implying that party 1 may secure $\max_{k \leq n} w^k(\mu)$, which is equal to $w(\mu)$ when $\mu \in (\mu^{n+1}, \mu^{n-1})$, hence a fortiori when $\mu \in (\widetilde{\mu}_{t-2}, \mu^*]$ (because $\widetilde{\mu}_{t-2} > \mu^{n+1}$).[39]

2. *Party 1's payoff is at most equal to $w_{t-2}(\mu)$ if $\mu_t = \mu_t^*$, and at most equal to $w(\mu)$ if $\mu_t > \mu_t^*$.*

Since party 1 cannot expect a share larger than v^* when he faces the rational party 2, party 1's equilibrium payoff (computed from date $t-2$) cannot be larger than $w_{t-2}(\mu)$ if $\mu_t = \mu_t^*$. For any offer Y_{t-2}, if $\mu_t > \mu_t^*$, the continuation equilibrium from date t is an equilibrium of the game without outside options. Party 1's equilibrium payoff (computed from date $t-2$) cannot be larger than the continuation equilibrium payoff obtained by party 1 in the game without outside option following an offer Y_{t-2}, hence it cannot be larger than $w(\mu)$.

[39] See footnote 38.

3. *If $Y_{t-2} < v^n$ and $Y_{t-2} \neq \delta v^*$, or if $Y_{t-2} = v^n$ and $\mu_t = \mu_t^*$, party 1 gets a payoff strictly smaller than $w_{t-2}(\mu)$.*

The payoff $w_{t-2}(\mu)$ may be obtained by party 1 if $Y_{t-2} = \delta v^*$ and $\mu_{t-1} = \mu_t = \mu_t^*$. Any offer $Y_{t-2} \in (\delta v^*, v^n)$ gives strictly less than $w_{t-2}(\mu)$ because it does not affect the probability that party 2 reveals herself (since $\mu_t = \mu_t^*$ by Lemma 7, c.), but only decreases the share obtained by party 1 in the event where party 2 accepts the offer; The argument is identical when $Y_{t-2} = v^n$ and $\mu_t = \mu_t^*$; Similarly, offers below δv^* give strictly less than $w_{t-2}(\mu)$ because they are rejected and in the event where party reveals herself party 1 only obtains a share $\delta v^* < v^* = 1 - \delta v^*$.

We may now conclude and check that the induction hypothesis holds at date $t-2$.

- From steps 1 and 2 above, it follows that party 1's equilibrium payoff is uniquely defined and equal to $\max\{w_{t-2}(\mu), w(\mu)\}$.

- If $\mu < \widetilde{\mu}_{t-2}$, then party 1's equilibrium payoff is equal to $w_{t-2}(\mu)$. From step 2, we must have $\mu_t = \mu_t^*$ (because $w(\mu) < w_{t-2}(\mu)$). This implies, by Lemma 7, d., that only offers no larger then v^n may be optimal. Thus from step 3, only $Y_{t-2} = \delta v^*$ may be optimal.

- If $\mu > \widetilde{\mu}_{t-2}$, $w(\mu) > w_{t-2}(\mu)$, hence party 1's equilibrium payoff is equal to $w(\mu)$ and from step 2, we must have $\mu_t > \mu_t^*$. By the induction hypothesis, from date t, the continuation equilibrium is an equilibrium of the game without outside options. Let Y_{t-2}^* be an equilibrium offer made by party 1 at $t-2$, and let $\sigma^* \mid_{Y_{t-2}^*}$ be the continuation equilibrium starting at $t-2$ with the offer Y_{t-2}^*. Incentive constraints facing party 2 and party 1 in the continuation game following the offer Y_{t-2}^* are identical to that of the game without outside options. Since $\sigma^* \mid_{Y_{t-2}^*}$ gives $w(\mu)$ to party 1, $\sigma^* \mid_{Y_{t-2}^*}$ must be an equilibrium of the game without outside options.

- Finally, party 2 cannot obtain a payoff larger than the equilibrium payoff of the game without outside options because if it is optimal for party 1 to make an offer $Y_{t-2}^* > v^n$, then from d., $\mu_t > \mu_t^*$, and, as explained above, the continuation equilibrium (starting at $t-2$ with the offer Y_{t-2}^*) must be an equilibrium of the game without outside options.

Therefore the induction hypothesis holds at date $t-2$. ■

Proof of Lemma 6: Consider an equilibrium, and let μ_{t-1} denote the belief at date $t-1$ in the event where party 2 rejects Y_{t-2}. If $\mu_{t-1} < 1$, then at date $t-1$, party 2 must reveal himself with probability 1 (because otherwise he obtains a payoff at most equal to δv^* at date t, while he can get v^* immediately by revealing himself). It follows that at date $t-2$, party 2 strictly prefers to accept any offer $Y_{t-2} > \delta v^*$, and to reject any offer $Y_{t-2} < \delta v^*$. ■

Proof of Lemma 7: We first show that no matter what Y_{t-2} is, we must have $\mu_t \geq \mu_t^*$. Indeed, otherwise, $\mu_t < \mu_t^*$, and by definition of μ_t^*, we would have $\delta w_t(\mu_t) > \max\{1 -$

$\theta, \delta w(\mu_t)\}$. Hence party 1 would strictly prefer to reject the obstinate demand at $t-1$. By the induction hypothesis, and since $\mu_t^* \leq \widetilde{\mu}_t$, party 1 would make an offer δv^* at t, which party 2 would accept. But then party 2 would strictly prefer to reveal herself at $t-1$ rather than accepting δv^* at t (implying that $\mu_t = 1$, contradicting $\mu_t < \mu_t^*$).

c. Assume by contradiction that $Y_{t-2} < v^n$ and $\mu_t > \mu_t^*$. We will show that i) if $n = 0$ (that is, $\widetilde{\mu}_t \in [\mu^0, 1)$) then party 1 accepts the obstinate demand at $t-1$; ii) if $n \geq 1$ (that is, $\widetilde{\mu}_t < \mu^0$) then party 1 offers at least v^{n-1} at t. Note that if i) and ii) hold, then $\mu_{t-2} = \mu_t$ [since in both cases party 2 strictly prefers to reject any offer $Y_{t-2} < v^n$ and to offer θ next], which contradicts the fact that $\mu_t > \mu_t^* \geq \mu_{t-2}$. We now prove that i) and ii) hold.

By definition of $\widetilde{\mu}_t$ and μ_t^*, we either have $\widetilde{\mu}_t = \mu_t^* < \mu^0$ or $\mu^0 \leq \mu_t^* \leq \widetilde{\mu}_t$.[40] Therefore i) if $n = 0$, then (by definition of n) $\mu^0 \leq \widetilde{\mu}_t$, hence $\mu^0 \leq \mu_t^* \leq \widetilde{\mu}_t$. Since $\mu_t > \mu_t^*$ and since $1-\theta = \delta w(\mu^0)$, we conclude that at date $t-1$, party 1 strictly prefers to accept the obstinate offer $1-\theta$; ii) if $n = 1$, then (by definition of n) $\widetilde{\mu}_t < \mu^0$, hence $\mu_t^* = \widetilde{\mu}_t$. Thus $\mu_t > \widetilde{\mu}_t$, and since Proposition 13 holds from date t on, the continuation equilibrium is an equilibrium of the game without outside options. Since $\mu_t > \widetilde{\mu}_t \geq \mu^n$, party 1 offers at least v^{n-1} by Proposition 10.

b. Party 2 strictly prefers to accept $Y_{t-2} > \delta v^*$ at $t-2$ rather than revealing herself at date $t-1$. So we must have $\mu_t = \mu_{t-1}$.

d. By step b., we have $\mu_t = \mu_{t-1}$. To prove $\mu_t > \mu_t^*$, we distinguish two cases again. i) $n = 0$. Then $Y_{t-2} > v^0 = \delta\theta$, so the offer is accepted by party 2 if rational, and $\mu_{t-1} = \mu_t = 1$. ii) $n \geq 1$. Assume by contradiction that $\mu_t = \mu_t^*$, then by definition of n, and since $\mu_t^* \leq \widetilde{\mu}_t$, we have $\mu_t < \mu^{n-1}$. Since Proposition 13 holds from date t on, party 2's equilibrium payoff computed from date t would be at most equal to v^{n-1} (by Proposition 10). Hence party 2 would strictly prefer to accept Y_{t-2} right away (contradicting $\mu_t = \mu_t^* < 1$).

a. Any offer $Y_{t-2} < \delta v^*$ is rejected because party 2 strictly prefers to reveal herself at date $t-1$. So we must have $\mu_{t-1} = \mu_{t-2}$. Besides, step c. implies $\mu_t = \mu_t^*$. ■

A Perfect Bayesian Nash equilibrium of the game with delayed outside options. Proposition 13 has derived conditions that any perfect Bayesian equilibrium must satisfy. For example, it uniquely defines party 1 and 2's behavior at any date t, when it is party 1's turn to make an offer and $\mu^t < \widetilde{\mu}_t$. Our objective in what follows is to provide a complete description of a perfect Bayesian equilibrium. The main difficulty will be to define the probability with which party 1 offers δv^* at date t when his current belief is equal to the threshold $\widetilde{\mu}_t$. It will have to be set so that at date $t-2$, party 2 is indifferent between accepting δv^* and rejecting

[40] Recall the definitions of μ^0, μ_t^* and $\widetilde{\mu}_t$ for $t < \tau$. We have $\delta w(\mu^0) = 1-\theta$, $\delta w_t(\mu_t^*) = \max\{1-\theta, \delta w(\mu_t^*)\}$ and $w(\widetilde{\mu}_t) = w(\mu_t^*)$.

the offer.

As for the description of the equilibrium of the game without outside options, our candidate equilibrium strategy profile for the game with delayed outside options is described by considering at any date, at any node and for any current belief $\mu > 0$ about the obstinacy of party 2 that party 1 holds *at that node*, the behavioral strategies of parties 1 and 2 (induced by the strategy profile).

It will be convenient to let σ_μ^* denote the strategy profile constructed from σ^* (the equilibrium strategy profile of the game without outside options defined in Appendix A) at a date where party 1 is the proposer, the current belief is μ and the history of moves is empty. Also, between any two dates where party 1 is the proposer, there are four nodes depending on who proposes an offer or responds to an offer. We describe these four possible nodes by $z \in \{1P, 2R, 2P, 1R\}$ (where for example $2P$ stands for the node where party 2 is the proposer) and let $\sigma^*_{\mu,Y^-,z}$ denote the strategy profile constructed from σ^* at a node z where the current belief is μ and the last offer of party 1 is Y^-.

We also denote by $Y^*(\mu)$ the first offer made by party 1 according to the equilibrium strategy profile σ_μ^*, and recall that $w(\mu)$ denotes the payoff obtained by party 1 under σ_μ^*.

We let Y_t denote the offer made by party 1 at t. We also let $\bar{t}$ be the first (earliest) date t for which $\mu^*_{t+2} > \mu^0$. (Thus for any date $t < \bar{t}$, we have $\mu^*_{t+2} = \widetilde{\mu}_{t+2}$).[41] For any date $t \leq \tau$, we define a threshold offer Y_t^* as follows:

$$
\begin{aligned}
Y_t^* &= \delta^2 Y^*(\mu^*_{t+2}) \text{ if } t < \bar{t} \\
Y_t^* &= \delta\theta \text{ if } t \geq \bar{t}.
\end{aligned}
$$

The interpretation of Y_t^* will be as follows. In equilibrium, as soon as an offer $Y_t \geq Y_t^*$ is made, the continuation game is as if party 1 had no outside options. When no offers above such threshold has been made, and if the current date t belief is no larger than $\widetilde{\mu}_t$, the outside option matters, and party 1 offers δv^* with positive probability.

I. *Consider a date t at which party 1 is the proposer.* *When $t > \tau$, party 1 offers δv^*; party 2 if rational accepts the offer Y_t made by party 1 at t if and only if $Y_t \geq \delta v^*$. When $t \leq \tau$:*

a) Party 1's offer:

At date 1, party 1 offers δv^*. At any later date:[42]

[41] Recall that μ^0 is such that $\delta w^0(\mu^0) = 1 - \theta$.

[42] *Note that we do not indicate party 1's behavior for all possible pairs (μ, Y_{t-2}). This is because, given party 2's behavior, a rejection of the offer Y_{t-2} imposes constraints on party 1's belief. Similarly, on paths where party 2 mimicks the obstinate type, party 1's belief at any date $t \geq \bar{t}$ may never be equal to $\widetilde{\mu}_t$ (either $Y_{t-2} < Y^*_{t-2} (= \delta\theta)$, and then $\mu_t = \mu_t^*$, or $Y_{t-2} \geq Y^*_{t-2}$, and then $\mu_t = 1$).*

- If $\mu > \widetilde{\mu}_t$ or $Y_{t-2} \geq Y^*_{t-2}$: party 1 follows $\sigma^*_{\mu,Y_{t-2},1P}$;
- If $\mu = \widetilde{\mu}_t$, $Y_{t-2} < Y^*_{t-2}$ and $t < \bar{t}$, party 1 offers δv^* with probability $q(Y_{t-2})$, and $Y^*(\mu)$ with probability $1 - q(Y_{t-2})$, where $q(Y_{t-2})$ satisfies:

$$Y_{t-2} = q(Y_{t-2})\delta^3 v^* + (1 - q(Y_{t-2}))\delta^2 Y^*(\mu)$$

if $Y_{t-2} \in [\delta v^*, Y^*_{t-2})$, and we choose $q(Y_{t-2}) = q(\delta v^*)$ if $Y_{t-2} < \delta v^*$.
- If $\mu < \widetilde{\mu}_t$, party 1 offers δv^*;
b) Party 2's response to (date t) offer Y_t:
- If $Y_t \geq Y^*_t$, party 2 follows $\sigma^*_{\mu,Y_t,2R}$.
- If $Y_t \in [\delta v^*, Y^*_t)$, party 2 accepts the offer with probability $\max\{1 - \mu/\mu^*_{t+2}, 0\}$.
- If $Y_t < \delta v^*$, party 2 rejects the offer.

II. Consider a date $t+1$ in which party 2 is the proposer. When $t+1 > \tau$, then party 2 offers δv^* to party 1, and party 1 accepts any offer $X \geq \delta v^*$, rejects any offer $X < \delta v^*$ (and then opts out if $t \geq T$). When $t+1 \leq \tau$, we have:
a) party 2's offer:
- If $\mu > \mu^*_{t+2}$, party 2 offers $1 - \theta$.
- If $\mu \leq \mu^*_{t+2}$, party 2 offers δv^* to party 1 with probability $1-\mu/\mu^*_{t+2}$ and $1-\theta$ with probability μ/μ^*_{t+2}.
b) party 1's response: After any offer $X \neq 1-\theta$, the current belief becomes $\mu = 0$ and continuation play follows that described in Proposition 1. After the obstinate offer $X = 1-\theta$:
- If $\mu > \mu^*_{t+2}$, party 1 follows $\sigma^*_{\mu,Y_t,1R}$.
- Otherwise, party 1 rejects the offer if $t+2 < \bar{t}$; or party 1 rejects it with probability $\overline{q}(Y_t)$ if $t+2 \geq \bar{t}$, where $\overline{q}(Y_t)$ satisfies:

$$Y_t = \overline{q}(Y_t)\delta^3 v^* + (1 - \overline{q}(Y_t))\delta\theta$$

if $Y_t \in [\delta v^*, Y^*_t)$, and where $\overline{q}(Y_t) = \overline{q}(\delta v^*)$ if $Y_t < \delta v^*$.

Under the proposed strategy profile, if initially the prior probability that party 2 is obstinate is given by $\varepsilon < \widetilde{\mu}_1$, then party 1 starts by offering δv^* which is accepted with probability $1 - \frac{\varepsilon}{\mu^*_3}$. Then starts a phase where parties 1 and 2 play a war of attrition: at any date, party 1 insists on the Rubinstein partition with probability $q(\delta v^*)$ (very close to 1), while party 2 accepts with probability equal to $1 - \frac{\mu^*_t}{\mu^*_{t+2}}$ (very close to 1 too). The sequence of beliefs that party 1 holds at dates where he is the proposer (odd dates), along a path where party 2 mimics the obstinate type and party 1 insists on the Rubinstein partition, is:

$$\varepsilon, \mu^*_3, \mu^*_5, ..., \mu^*_\tau.$$

And if party 2 rejects party 1's offer at τ, then party 2 is believed to be obstinate with probability 1; party 1 then waits for the outside option.

Also, at any date t along the path described above, and in the event where party 1 does not insist on the Rubinstein partition, he offers $Y^*(\mu_t^*)$. Since $Y^*(\mu_t^*)$ is no smaller than Y_t^* by construction,[43] continuation play then follows $\sigma^*_{\mu,Y^*(\mu_t^*),2R}$, that is, continuation play follows that of the game without outside options.

The following Proposition states conditions under which the above strategy profile is a Perfect Bayesian Equilibrium of the game with delayed outside options.

Proposition 14 *Define N and μ^N as in (5-10), and μ_t^* as in (33). Assume that $\mu_3^* > \mu^N$. Then the above strategy profile defines a Perfect Bayesian Equilibrium of the game with delayed outside option.*

Note that Lemma 5 implies that for any fixed $\underline{v}_1$, the conditions of Proposition 14 are satisfied for δ close enough to 1.

Proof of Proposition 14: We briefly check incentives for $t \leq \tau$ (for $t > \tau$ these are straightforward).

Party 2's incentives (sketch): The probabilities $q(.)$ and $\overline{q}(.)$ are precisely chosen so that after any history of offers leading to a current date t belief below μ^*_{t+2}: a) if the current date t offer Y_t of party 1 lies in $[\delta v^*, Y_t^*)$, then party 2 when rational is indifferent between accepting and rejecting Y_t; and b) if the current date t offer Y_t is strictly below δv^*, party 2 when rational strictly prefers to reject Y_t and then is indifferent between offering δv^* and $1-\theta$ to party 1. Indeed, if party 2 rejects, then the current belief becomes equal to μ^*_{t+2}. Party 2 then offers the obstinate demand. If $t+2 \geq \bar{t}$, then party 2 gets δv^* with probability $\overline{q}(Y_t)$ at date $t+2$, and θ at date $t+1$ otherwise; if $t+2 < \bar{t}$, then party 1 rejects the obstinate demand at $t+1$, and at date $t+2$, party 2 gets δv^* with probability $q(Y_t)$ and $Y^*(\mu^*_{t+2}) = \frac{Y_t^*}{\delta^2}$ with probability $1 - q(Y_t)$.

After any offer $Y \geq Y_t^*$, the proposed behavior of party 2 is to follow $\sigma^*_{\mu,Y,2R}$, that is, to accept the offer Y with probability equal to $1 - \frac{\mu}{\mu^{k-1}}$ (where k is set so that $Y \in [v^k, v^{k-1})$) if $\mu < \mu^{k-1}$ (and to reject the offer Y otherwise). Let n be such that $\mu^*_{t+2} \in (\mu^n, \mu^{n-1}]$, and let us focus on the case where $n \geq 1$.[44] Then $Y^*(\mu^*_{t+2}) = v^{n-1}$, hence $Y_t^* = v^n$. Thus either $\mu \geq \mu^{n-1}$, or $\mu < \mu^{n-1}$ and any offer $Y \geq Y_t^*$, if rejected, leads to a current date $t+1$ belief at least equal to μ^{n-1}. In both cases, the current belief thus becomes at least equal to μ^{n-1}, and

[43] Indeed, either $t \geq \bar{t}$, in which case $\mu_t^* > \mu^0$ and $Y^*(\mu_t^*) = \delta\theta = Y_t^*$, or $t < \bar{t}$, in which case $Y^*(\mu_t^*) \geq \delta^2 Y^*(\mu^*_{t+2}) = Y_t^*$ because $\mu^{n+1} < \mu_t^* < \mu^*_{t+2} < \mu^{n-1}$ for some n (see footnote 38).

[44] In case $n = 0$, that is, in case $\mu^*_{t+2} > \mu^0$, then $Y_t^* = \delta\theta$. If $Y \geq Y_t^*$, following $\sigma^*_{\mu,Y,2R}$ requires that party 2 if rational accepts Y. This is clearly optimal for party 2.

since μ^{n-1} is at least equal to μ^*_{t+2}, continuation play from that date thus coincides with that of the game without outside options (according to the proposed strategy profile). Following $\sigma^*_{\mu,Y,2R}$ is thus optimal for party 2, as in proposition 11 (the conditions of Proposition 11 are satisfied because after party 2's rejection the current belief μ is at least equal to μ^*_3, so under the conditions of Proposition 14, there exists $n \leq N$ such that $\mu \in (\mu^n, \mu^{n-1}])$.

Party 1's incentives (sketch): At date $t+1$ after the obstinate offer of party 2, the current belief is equal to μ^*_{t+2}. If $t+2 < \bar{t}$, then $\delta w_{t+2}(\mu^*_{t+2}) \geq 1-\theta$ and it is therefore optimal for party 1 to reject the obstinate offer. If $t+2 \geq \bar{t}$, then by construction $\delta w_{t+2}(\mu^*_{t+2}) = 1-\theta$, hence party 1 is indifferent between accepting and rejecting party 2's obstinate offer.

At date t, any offer $Y \in [\delta v^*, \delta^2 Y^*(\mu^*_{t+2}))$ yields the same continuation behavior from party 2. Choosing δv^* among these offers is thus optimal. Party 1 obtains $w_t(\mu)$ by doing so, which he has to compare to what he obtains when he follows $\sigma^*_{\mu,Y_{t-2},1P}$, that is, $w(\mu)$. By definition of $\widetilde{\mu}^t$, which option is preferred depends on how μ compares with $\widetilde{\mu}^t$. ■