

Type Indeterminacy: A Model of the KT(Kahneman-Tversky)-man

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Abstract

In this note we propose to use the mathematical formalism of Quantum Mechanics to capture the idea that agents' preferences, in addition to being typically uncertain, can also be **indeterminate**. They are determined (realized, and not merely revealed) only when the action takes place. An agent is described by a **state** which is a superposition of potential types (or preferences or behaviors). This superposed state is projected (or "collapses") onto one of the possible behaviors at the time of the interaction. In addition to the main goal of modelling uncertainty of preferences which is not due to lack of information, this formalism, seems to be adequate to describe widely observed phenomena like framing and instances of noncommutativity in patterns of behavior. We propose two experiments to test the theory.

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1 Introduction

Recently, models of quantum games have been proposed to study in which respect the extension of classical moves to quantum ones (i.e. complex linear combination of classical moves) could affect the analysis of the game. For example, it has been shown (Eisert et al. 1999) that allowing the players to use quantum strategies in the Prisoners' Dilemma, is a way to escape the well-known 'bad feature' of this game.¹ From a game theoretical point of view the approach consists of changing the strategy spaces and thus the interest of the results rests on the appeal of these changes.²

This paper also proposes to use the mathematical formalism of Quantum Mechanics but with a different intention; the goal being to model uncertain preferences. The basic idea is that the Hilbert space model of Quantum Mechanics can be thought of as a very general contextual predictive tool particularly well suited to describe experiments in psychology or in 'revealing' preferences.

The well established Bayesian approach suggested by Harsanyi to model incomplete information consists of a chance move that selects the actual types of the players and informs each player of his own type. For the purpose of this note, we underline the following essential implication of this approach: all uncertainty about a player's type reflects exclusively others' incomplete knowledge about it. This follows from the fact that a Harsanyi type is fully determined. It is a well-defined characteristic of a player which is known to him. Consequently, from the point of view of other players, uncertainty about the type can only be due to lack of *information*. Each player has a probability distribution over the type of the other players, but his own type is fully determined and is known to him.

This brings us to the first important point at which we depart from the classical approach: we propose that in addition to informational reasons, the uncertainty concerning preferences is due to *indeterminacy*: prior to the moment a player acts, his (behavior) type is indeterminate. The 'state', representing the player, is a 'superposition' of potential types. It is only at

¹In the classical version of the dilemma, the dominant strategy for both players is to defect and thereby to do worse than if they would both decide to cooperate. In the quantum version, there is a couple of quantum strategies which is both a Nash equilibrium and Pareto optimal and whose payoff is that of joint cooperation.

²This approach is closely related to quantum computing. It makes use of the notion of q-bits (quantum units of information) and of that of measurement.

the moment when the player selects a choice, that a specific type is realized. It is not merely revealed but rather ‘determined’, in the sense that prior to the choice, there is an irreducible richness of potential types. Thus we suggest that in modelling a decision situation, we do not assume that the preferences characteristics can always be fully known with certainty (not to the decision maker and even not to the analyst). Instead, what is known is the *state* of the agent: a vector in a Hilbert space spanned by types which encapsulates all idiosyncratic information needed to predict how the agent is expected to behave in different decision situations.

This idea, daringly borrowed from Quantum Mechanics to the context of decision and game theory, is very much in line with Tversky and Simonson according to which ” *There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed - not merely revealed - in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice.*” (in Kahneman and Tversky 2000). This view is also consistent with that of cognitive psychology that teaches to distinguish objective reality from the proximal stimulus to which the observer is exposed, and to distinguish both reality and the stimulus from the mental representation of the situation that the observer eventually constructs. More generally this view fits with the observation that players (even highly rational) may act differently in game theoretically equivalent situations which differ only by seemingly irrelevant aspects (framing, prior unrelated events etc.). Our theory for why people act differently in game theoretical equivalent situations is that they may not be representable by the same state: (revealed) preferences are contextual because of intrinsic indeterminacy.

The basic analogy with Physics, which makes it appealing to borrow the mathematical formalism of Quantum Mechanics to social sciences, is the following: We view observed play, decisions and choices as something similar to the result of a *measurement* (a measurement of the player’s state of preferences or the player’s type). The decision situation is modeled as an operator (called *observable*), and the resulting behavior as an eigenvalue of that operator. The analogy to the non-commutativity of observables (a very central feature of Quantum Mechanics) is, in many empirically observed phenomena, like the following one. Suppose that we subject a given agent to the same decision situation in two different contexts (the contexts may vary with respect to the decision situations that precede the investigated one, or it can vary with respect to the framing in the presentation of the decision, (cf.

Selten 1998)). If we do not observe the same decision in the two contexts, then the classical approach is to say that the two decision situations *are not the same*; they should be modeled so as to incorporate the context. In many cases however such an assumption i.e. that the situations are different, is difficult to justify. And so the standard theory leaves a host of behavioral phenomena unexplained: ‘the behavioral anomalies’ (cf. McFadden 1999).

In contrast, we propose to say that the observed decisions *were not taken by an agent of the same type*. The context e.g. a past decision, is viewed as an operator that does not commute with the operator associated with the investigated decision situation; its operation on the player has changed the state (the type) of the agent. As in Quantum Mechanics the phenomenon of non-commutativity of decision situations (measurements) leads us to make a hypothesis that an agent’s preferences or a *preference system* is represented by a state which is indeterminate and gets determined (with respect to any particular type characteristics) in the process of interaction with the environment.

The objective of this paper is to propose a theoretical framework that extends the Bayes-Harsanyi model to accommodate various forms of the so-called behavioral anomalies. Adopting McFadden’s terminology, we attempt to provide a model for the KT(Kahneman-Tversky)-man as opposed to what he calls the Chicago man (McFadden 1999). Our work is related to Random Utility Models (RUM) as well as to Behavioral economics. RUM models have provided very useful tools for explaining and predicting deviations from standard utility models. RUM, however cannot accommodate the kind of drastic and systematic deviations characteristic of the KT man. These models rest on an hypothesis of ‘consumer sovereignty’ i.e. ‘the primacy of desirability over availability’, and they assume stable taste templates (see McFadden 2000 for a very good survey). In RUM preferences are non-contextual by construction while the proposed Hilbert Space Model (HSM) of preferences is a model of contextual preferences. Behavioral economics have contributed a variety of explanations for deviations from self-interest (see Camerer 1997). Most often they elaborate on the preference function itself introducing for instance ‘trade off contrast’, ‘extremeness aversion’ (Kahneman and Tversky ed. 2000), ‘fairness concerns’ (Rabin 1993) or ‘self-control costs’ (Gul and Pesendorfer 2001)³. The explanations proposed by behavioral economics are

³Gul and Pesendorfer’s work also addresses deep conceptual issues related to the specific class of behavioral anomalies they study.

often specific to the deviation that is being explained. Yet, other explanations would appeal to bounded rationality e.g. "superficial reasoning" (Selten 1998). In contrast, the HSM is a framework model that addresses structural properties of preferences.

In section 2, the framework is presented and basic notions of quantum theory are introduced. In Section 3, we propose applications to Social Sciences. We compare the predictions and interpretations of the HSM with those of probabilistic models. Two experiments to test the theory are suggested. An application of the approach to framing effects is proposed. Concluding remarks are gathered in the last section.

2 The basic framework

In this section we present the basic notions for our framework. They are heavily inspired by the mathematical formalism of Quantum Mechanics (see e.g. Cohen-Tannundji, Dui and Laloe,1973) from which we also borrow the notation.

2.1 The notion of *state*

The object of our investigation is the choice behavior which we interpret as revelation of the preferences of an agent in what we call a *Decision Situation* (DS). In this paper we focus on non-repeated non-strategic decision situations. This can be the choice between buying a Toshiba laptop at price x or a Compaq laptop at price y . The choice between a sure gain of \$100 or a bet with probability 0.5 to win \$200 and 0.5 to win \$0, is a DS as well as the choice between investing in a project or not, etc. When considering games we review them as decision situations from the point of view of a single player⁴. An agent or a 'preference system' is represented by a '*state*' which captures all the agent's potential behavior in the decision situations under consideration. Mathematically, a state $|\psi\rangle$ is a vector in a Hilbert space \mathcal{H} over the field of the complex numbers \mathcal{C} . This space has to be large enough to accommodate all the decision situations we want to study. The relationship between \mathcal{H} and the decision situations will be specified later.

⁴All information (beliefs) and strategic considerations are embedded in the definition of the 'choices'. Thus the agent's play of 'C' is a play of 'C' given e.g. his information (knowledge) about the opponent.

For technical reasons related to the probabilistic content of the state, it has to be of length one that is, $|\langle\psi|\psi\rangle|^2 = 1$. So all vectors of the form $\lambda|\psi\rangle$ where $\lambda \in \mathcal{C}$ correspond to the same state which we represent by such vector of length one. For this reason we shall consider only linear combinations in \mathcal{H} of the form $|\psi\rangle = \sum_{i=1}^n \lambda_i |\varphi_i\rangle$ with $\sum_{i=1}^n |\lambda_i|^2 = 1$ (such a linear combination is called a *superposition*). If $|\varphi_1\rangle \dots |\varphi_n\rangle$ are all states (i.e. unit vectors) then the superposition $|\psi\rangle$ is also a state.

2.2 The notions of *observable* and *measurement*

A decision situation is defined by the set of alternative choices available to the agent. When an agent selects a choice out of a set corresponding to a decision situation, we say that he ‘plays’ the DS. Each choice corresponds to a type in the relevant DS. To every Decision Situation A , we associate an *observable*, namely a specific Hermitian operator on \mathcal{H} which, for notational simplicity, we denote also by A .⁵ If A is the only decision situation we consider, we can assume that its eigenvectors which we denote by $|1_A\rangle, |2_A\rangle, \dots, |n_A\rangle$ all correspond to different eigenvalues, denoted by $1_A, 2_A, \dots, n_A$ respectively.

$$A|k_A\rangle = k_A|k_A\rangle, \quad k = 1, \dots, n.$$

As A is Hermitian, there is an orthonormal basis of the relevant Hilbert space \mathcal{H} formed with its eigenvectors. The basis $\{|1_A\rangle, |2_A\rangle, \dots, |n_A\rangle\}$ is the unique orthonormal basis of \mathcal{H} consisting of eigenvectors of A . It is thus possible to represent the agent’s state as a superposition of the vectors of this basis:

$$|\psi\rangle = \sum_{k=1}^n \lambda_k |k_A\rangle, \quad (1)$$

where $\lambda_k \in \mathcal{C}, \forall k \in \{1, \dots, n\}$ and $\sum_{k=1}^n |\lambda_k|^2 = 1$.

The expansion (1) can also be written as

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, \dots, \oplus \mathcal{H}_{n_A} \quad (2)$$

where \oplus denotes the direct sum of the subspaces $\mathcal{H}_{1_A}, \dots, \mathcal{H}_{n_A}$ spanned by $|1_A\rangle, \dots, |n_A\rangle$ respectively.⁶ Or, in terms of the projection operators

⁵An operator A on \mathcal{H} is Hermitian if $A' = A$ where A' is the adjoint (Hermitian conjugate) defined by $\langle\varphi|A'|\psi\rangle = \langle\psi|A|\varphi\rangle^*$ for all φ and ψ in \mathcal{H} .

⁶That is, for $i \neq j$ any vector in \mathcal{H}_{i_A} is orthogonal to any vector in \mathcal{H}_{j_A} and any vector in \mathcal{H} is a sum of n vectors, one in each component space.

$P_{1_A} + \dots + P_{n_A} = I_{\mathcal{H}}$ where P_{i_A} is the projection operators on \mathcal{H}_{i_A} and $I_{\mathcal{H}}$ is the identity operator on \mathcal{H} . This presentation will prove to be convenient when treating more than one observable. In fact if we want to treat both observables A with eigenvalues $\{1_A, 2_A, \dots, n_A\}$, and B with eigenvalues $\{1_B, 2_B, \dots, m_B\}$, we express the Hilbert space \mathcal{H} as a direct sum in two different ways:

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, \dots, \oplus \mathcal{H}_{n_A} \text{ and } \mathcal{H} = \mathcal{H}_{1_B} \oplus, \dots, \oplus \mathcal{H}_{m_B}$$

where \mathcal{H}_{i_A} is the subspace of eigenvectors of A with eigenvalue i_A (similarly of \mathcal{H}_{j_B}). Note however that in this case \mathcal{H}_{i_A} and \mathcal{H}_{j_B} will not in general be one dimensional. In the general case, when the subspaces \mathcal{H}_{i_A} are not one dimensional (i.e. when the eigenvalues i_A are degenerated) the superposition in eq. (1) still exists and is unique for each $|\psi\rangle$ but the vectors $|k_A\rangle \in \mathcal{H}_{k_A}$, may depend on $|\psi\rangle$.⁷

The realization of a type for an agent in the state $|\psi\rangle$, in a decision situation A is the actual choice of a particular action among all the possible actions in that situation. It is represented by a *measurement* of the observable A associated with this decision situation. According to the so-called *Reduction Principle* (see appendix), the result of such a measurement can only be one of the n eigenvalues of A . If the result is m_A i.e. the player selected the choice m_A , the superposition $\sum \lambda_i |i_A\rangle$ ‘‘collapses’’ onto the eigenvector associated with the eigenvalue m_A (i.e. the initial state $|\psi\rangle$ is projected onto the the subspace \mathcal{H}_{m_A} of eigenvectors of A with eigenvalue m_A). The probability that the measurement yields the result m_A is equal to $|\langle m_A | \psi \rangle|^2 = |\lambda_m|^2$ (where $\langle \cdot | \cdot \rangle$ denotes the inner product in \mathcal{H})⁸. As usual, we will interpret the probability of m_A either as the probability that one agent in the state $|\psi\rangle$ selects the choice m_A or as the proportion of the agents who will make the choice m_A in a population of many agents, all in the state $|\psi\rangle$.

Remark: Clearly, when only one DS is considered, the above description is equivalent to the traditional probabilistic representation of an agent by a probability vector $(\alpha_1, \dots, \alpha_n)$ in which α_k is the probability that the agent will choose action k_A . A distinction however is that in the HSM the probability

⁷To see that such expansion exists and is unique take a basis of \mathcal{H} which consists of basis of the component subspaces \mathcal{H}_{k_A} , expand $|\psi\rangle$ in this basis and let $|k_A\rangle$ be the part of the sum in \mathcal{H}_{k_A} of that expansion. This $|k_A\rangle$ is uniquely defined for each $|\psi\rangle$.

⁸We here for simplicity assume that all eigenvalues are ‘non degenerated’. We return to that issue below.

distribution is not directly given by the coefficients of the linear combination but by the square of their module. The advantage of the proposed formalism is in studying several decision situations and the interaction between them.

2.3 More than one Decision Situation

When studying more than one DS, say A and B , it turns out that a key question is whether the corresponding observables are commuting operators in \mathcal{H} i.e. whether $AB = BA$. The question whether two DSs can or cannot be represented by two commuting operators is an empirical question on which we comment later, we first study the mathematical implications of this feature.

2.3.1 Commuting Decision Situations

Let A and B be two DSs. If the corresponding observables commute then there is an orthonormal basis of the relevant Hilbert space \mathcal{H} formed by eigenvectors common to both A and B . Denote by $|i\rangle$ (for $i = 1, \dots, n$) these basis vectors. We have

$$A|i\rangle = i_A|i\rangle \text{ and } B|i\rangle = i_B|i\rangle.$$

In general the eigenvalues can be degenerated (i.e. for some i and j , $i_A = j_A$ or $i_B = j_B$). Any normalized vector $|\psi\rangle$ of \mathcal{H} can be written in this basis:

$$|\psi\rangle = \sum_i \lambda_i |i\rangle$$

where $\lambda_i \in \mathcal{C}$, and $\sum_i |\lambda_i|^2 = 1$. If we measure A first, we observe eigenvalue i_A with probability:

$$p_A(i_A) = \sum_{j:j_A=i_A} |\lambda_j|^2 \quad (3)$$

If we measure B first, we observe eigenvalue j_B with probability:

$$p_B(j_B) = \sum_{k:k_B=j_B} |\lambda_k|^2 \quad (4)$$

After having measured B and obtained the result j_B , the state $|\psi\rangle$ is projected into the eigensubspace \mathcal{E}_{j_B} spanned by the eigenvectors of B associated with j_B . More specifically, it collapses onto the state:

$$|\psi_{j_B}\rangle = \frac{1}{\sqrt{\sum_{k;k_B=j_B} |\lambda_k|^2}} \sum_{k;k_B=j_B} \lambda_k |k\rangle$$

(the factor $\frac{1}{\sqrt{\sum_{k;k_B=j_B} |\lambda_k|^2}}$ is necessary to make $|\psi_{j_B}\rangle$ a unit vector).

Now when we measure A on the agent in the state $|\psi_{j_B}\rangle$, we obtain i_A with probability

$$p_A(i_A|j_B) = \frac{1}{\sum_{k;k_B=j_B} |\lambda_k|^2} \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} |\lambda_k|^2$$

So when measuring B first and then A , the probability to observe the eigenvalue i_A is:

$$\begin{aligned} p_{AB}(i_A) &= \sum_j p_B(j_B) p_A(i_A|j_B) \\ &= \sum_j \frac{1}{\sum_{k;k_B=j_B} |\lambda_k|^2} \sum_{k;k_B=j_B} |\lambda_k|^2 \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} |\lambda_l|^2 \\ &= \sum_j \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} |\lambda_l|^2 = \sum_{l;l_A=i_A} |\lambda_l|^2 \\ &= p_A(i_A) \end{aligned}$$

Hence, $p_{AB}(i_A) = p_A(i_A)$, $\forall i$, and similarly $p_{BA}(j_B) = p_B(j_B)$, $\forall j$.

This expresses the fact that there is no ‘‘interference’’ between the two observations: we have the same probability distribution on the result of the measurement of A whether we measure A directly or measure B first and then measure A . The measurement of B does not perturb the measurement of A . In this case it is meaningful to speak of measuring A and B simultaneously. Whether we measure first A and then B or first B and then A , we have the same probability distribution on the joint outcome namely $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} |\lambda_k|^2$. In other words ‘ (A, B) have definite pair of values (i_A, j_B) ’ is a well defined event.

Remark: Note that again, as in the case of one DS, for such two DSs our model is equivalent to a standard (discrete) probability space in which the elementary events are $\{(i_A, j_B)\}$ and $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} |\lambda_k|^2$. In particular, in accordance with the calculus of probability we see that the conditional probability formula holds:

$$p_{AB}(i_A \wedge j_B) = p_A(i_A) p_B(j_B|i_A).$$

As an illustration of an example of this kind, consider the following two decision problems: Let A be the decision situation of choosing between a week vacation in Tunisia and a week vacation in Italy. And let B be the choice between buying 1000 euros of shares in Bouygues telecom or in Deutsche telecom. If the corresponding observables commute, we expect that a decision on the portfolio (B) does not affect decision-making regarding the location for vacation (A). And thus the order in which the decisions are made does not matter. It is quite plausible that A and B commute but whether or not this is in fact the case, is of course an empirical question.

Note that the commutativity of the observables does not exclude statistical correlations between the two observations. To see this consider the following example in which A and B each have two degenerated eigenvalues in a four dimensional Hilbert space. Denote $|i_A j_B\rangle$ ($i = 1, 2, j = 1, 2$) the eigenvector associated with eigenvalues i_A of A and j_B of B and let the state $|\psi\rangle$ be given by:

$$|\psi\rangle = \sqrt{\frac{3}{8}}|1_A 1_B\rangle + \sqrt{\frac{1}{8}}|1_A 2_B\rangle + \sqrt{\frac{1}{8}}|2_A 1_B\rangle + \sqrt{\frac{3}{8}}|2_A 2_B\rangle$$

Then

$$p_A(i_A|1_B) = \frac{\frac{3}{8}}{\frac{3}{8} + \frac{1}{8}} = \frac{3}{4} \quad (\text{for } i = 1 \text{ or } i = 2)$$

So if we first measure B and find say 1_B , it is much more likely (prob. $\frac{3}{4}$) that measuring A we find 1_A rather than 2_A (prob. $\frac{1}{4}$). In other words the statistical correlation is informational: information about one of the observable is relevant to the prediction of the value of the other. But the two *interactions* (measurements), do not perturb each other i.e., the distribution of the outcomes of the measurement of A is the same whether or not we measure B first.

2.3.2 Non commuting Decision Situations

It is when considering Decision Situations associated with observables that do not commute that the predictions of the HSM differ from those of the probabilistic model. As we shall see, the distinction is due to the presence of cross terms also called interferences. In the appendix we provide a brief description of the classical interferences experiment in Physics.

For the sake of simplicity, assume that two decision situations A and B have the same number n of possible choices, that means that the observables

A and B have the (non degenerated) eigenvalues $1_A, 2_A, \dots, n_A$ and $1_B, 2_B, \dots, n_B$ respectively and each of the sets of eigenvectors $\{|1_A\rangle, |2_A\rangle, \dots, |n_A\rangle\}$ and $\{|1_B\rangle, |2_B\rangle, \dots, |n_B\rangle\}$ is an orthonormal basis of the relevant Hilbert space. So each element of one basis, say $|j_B\rangle$, is a superposition of the elements of the other basis:

$$|j_B\rangle = \sum_{i=1}^n \mu_{ij} |i_A\rangle$$

Let $|\psi\rangle$ be the initial state of the player.

$$|\psi\rangle = \sum_{j=1}^n \lambda_j |j_B\rangle = \sum_{j=1}^n \sum_{i=1}^n \lambda_j \mu_{ij} |i_A\rangle$$

If the agent plays directly A , he will choose i_A with the probability $p_A(i_A) = \left| \sum_{j=1}^n \lambda_j \mu_{ij} \right|^2$. If he plays first B , he will select choice j_B with the probability $|\lambda_j|^2$ and will be projected in the state $|j_B\rangle$ then he will select choice i_A in the game A with the probability $|\mu_{ij}|^2$. So the probability of choice i_A after having played B is $p_{AB}(i_A) = \sum_{j=1}^n |\lambda_j|^2 |\mu_{ij}|^2$ which is in general different from $\left| \sum_{j=1}^n \lambda_j \mu_{ij} \right|^2$. That is: Playing B first, changes the way A is played. The difference stems from the so called *interference terms*:

$$\begin{aligned} p_A(i_A) &= \left| \sum_{j=1}^n \lambda_j \mu_{ij} \right|^2 = \sum_{j=1}^n |\lambda_j|^2 |\mu_{ij}|^2 + \underbrace{2 \sum_{j \neq j'} [\operatorname{Re} (\lambda_{j'} \mu_{ij}) (\lambda_j^* \mu_{ij'}^*)]}_{\text{Interference term}} \\ &= p_{AB}(i_A) + \text{interference term} \end{aligned}$$

The interference term is the sum of cross terms involving the coefficients of the superposition which are also called the *amplitudes* (of the probabilities).

Some helpful intuition about interference effects may be provided using the concept of ‘propensity’ (K. Popper’s (1992)). Imagine an agent’s mind as a system composed of a large variety of propensities to act (corresponding

to the different possible actions). As long as the agent is not required to choose his action, propensities coexist in his mind; The agent has not ‘made up his mind’. A decision situation forces him to determine himself. The DS operates on this state of ‘hesitation’ where the alternative propensities are present. The DS triggers the emergence of a single type (of behavior). The agent’s state of hesitation is replaced by a determinate preference. The cross terms express competition/reinforcement of propensities in this process, and they contribute to determining the probabilities for observing the ‘emergence’ of the realized type.

An illustration of an example of this kind may be supplied by the experiment reported in Knez and Camerer (2000). The two DSs studied are the Weak link coordination (WL) game and the Prisoner’s Dilemma (PD) game.⁹ They compare the distribution of choices in the Prisoner Dilemma (PD) game when it is preceded by a Weak Link (WL) game and when only the PD game is being played. Their results show that playing the WL game affects the play of individuals in the PD game. The explanation offered by the authors makes use of what they call a “transfer effect”.¹⁰ The precedent hypothesis is about shared expectations of cooperative play. This informational argument cannot explain the high rate of cooperation (37.5 %) in the last round of the PD game (table 5 p. 206) however. Instead, the experimental results are consistent with the hypothesis that the play of the WL game affected players’ preferences. We propose that the WL and the PD dilemma are two DS that do not commute. In such a case we do expect a difference in the distributions of choices in the PD depending on whether or not it was preceded by a play of the WL or another PD game. This distinction is due to interference effects (present in the second case) which is a feature of behavior indeterminacy.

Remark: In this case, A and B cannot have simultaneously a defined

⁹The weak link game is a type of coordination game in which each player picks an action from a set of integers. The payoffs are defined in such a manner that each player wants to select the minimum of the other players but everyone wants that minimum to be as high as possible.

¹⁰The transfer effect hypothesis is as follows: “the shared experience of playing the efficient equilibrium in the WL game creates a precedent of efficient play strong enough to (...) lead to cooperation in a finitely repeated PD game”, p.206, Knez and Camerer (2000).

value and there is no probability distribution on the event ‘to have the value i_A for A and the value j_B for B ’. The conditional probability formula does not hold any more.

$$p_A(i_A) \neq \sum_{j=1}^n p_B(j_B) p(i_A|j_B)$$

$$\text{Indeed, } \left| \sum_{j=1}^n \lambda_j \mu_{ij} \right|^2 = p_A(i_A) \neq \sum_{j=1}^n p_B(j_B) p(i_A|j_B) = \sum_{j=1}^n |\lambda_j|^2 |\mu_{ij}|^2.$$

In terms of our example above, the joint probability for event (i_A, j_B) which is to play i_A in the WL game and j_B in the PD cannot be defined in a single probability space. Playing first the WL game *does* affect the distribution of observed play in the PD.

3 Indeterminacy in Social Sciences

3.1 Context dependent preferences

As a first application let us consider a situation when a DS is played in a context. The context is an interaction that precedes the investigated DS.¹¹ Specifically consider the decision operator associated with the Prisoner’s Dilemma (PD) A with $\{|1_A\rangle, |2_A\rangle\}$ as eigenvectors corresponding to the choices cooperate (C) associated with eigenvalue 1_A and defect (D) associated with eigenvalue 2_A . The payoffs are given in the matrix below where the choices of the second player are put into brackets to indicate that we are considering only one agent (the row player) as a decision maker while the other (the column player) is part of the description of the DS.

	[C]	[D]
$1_A(C)$	3	4
$2_A(D)$	0	1
	4	1

Prisoner’s Dilemma

¹¹What we call ‘context’ here corresponds to what Tversky and Simonson call ‘background context’ p.521 in Kahneman and Tversky (2002)

It is possible to write $|\psi\rangle$ in this basis:

$$|\psi\rangle = \lambda_1 |1_A\rangle + \lambda_2 |2_A\rangle$$

Suppose that before playing the PD the agent has played another DS , a dictator game, DG,¹² (with two options) associated with an operator B that has $|1_B\rangle$, $|2_B\rangle$ as its eigenvectors corresponding to the actions “share fairly” (F) and “share selfishly” (S). The dictator game is played against a third player i.e. not involved in the PD game.

	For me	For him
1_B (F)	50	50
2_B (S)	95	5

Dictator Game

It is also possible to write $|\psi\rangle$ in this basis: $|\psi\rangle = \nu_1 |1_B\rangle + \nu_2 |2_B\rangle$. Assume that the two operators do not commute. We write the eigenvectors $\{|1_B\rangle, |2_B\rangle\}$ as superpositions of the basis $\{|1_A\rangle, |2_A\rangle\}$:

$$|1_B\rangle = \mu_{11} |1_A\rangle + \mu_{12} |2_A\rangle$$

$$|2_B\rangle = \mu_{21} |1_A\rangle + \mu_{22} |2_A\rangle$$

Hence:

$$|\psi\rangle = \lambda_1 |1_A\rangle + \lambda_2 |2_A\rangle = \nu_1 |1_B\rangle + \nu_2 |2_B\rangle \quad (5)$$

$$= (\nu_1 \mu_{11} + \nu_2 \mu_{21}) |1_A\rangle + (\nu_1 \mu_{12} + \nu_2 \mu_{22}) |2_A\rangle \quad (6)$$

The expression in (5) and (6) illustrates a central feature of the HSM formalism. Namely that any two DS associated with observables that do not commute, are linked by a basis transformation matrix S with $S_{ji} = \langle i_A | j_B \rangle = \mu_{ji}$. In our two dimensional example S writes

$$S = \begin{pmatrix} \langle 1_A | 1_B \rangle & \langle 2_A | 1_B \rangle \\ \langle 1_A | 2_B \rangle & \langle 2_A | 2_B \rangle \end{pmatrix}.$$

so we have

¹²The Dictator Game is a two players game in which one player, the Dictator, decides on a split of an amount of money, say 100 units, between him and the other player who is actually a dummy

$$|\psi\rangle = (\lambda_1 \lambda_2) \begin{pmatrix} |1_A\rangle \\ |2_A\rangle \end{pmatrix} = (\lambda_1 \lambda_2) S \begin{pmatrix} |1_B\rangle \\ |2_B\rangle \end{pmatrix}.$$

The basis transformation matrix that links any two non-commuting observables is of course *independent of* $|\psi\rangle$. The elements of this matrix do have an interpretation as (the square root of) the (statistical) correlations between the types of the two DSs. These invariants can, in principle, be estimated using empirical data and used to calibrate predictive models of decision behavior (cf experiment 2 below).

It follows from (6) that if the agent directly plays the PD, he will play the action i_A with the probability $p_A(i_A) = |(\nu_1\mu_{1i} + \nu_2\mu_{2i})|^2$. That means that both potential types (fair (1_B), selfish(2_B)) affect his behavior.

Playing the dictator game corresponds to the collapse of $|\psi\rangle$ onto $|1_B\rangle$ or $|2_B\rangle$. The probability for a collapse on $|k_B\rangle$ is $|\nu_k|^2$ and the probability that an agent of type $|b_k\rangle$ plays the action i_A is $|\mu_{ki}|^2$. Thus, when playing B first, the agent will play the action i_A with the probability $p_{AB}(i_A) = |\nu_1|^2|\mu_{1i}|^2 + |\nu_2|^2|\mu_{2i}|^2 \neq p_A(i_A)$. The probability for playing action i_A is not the same as when the agent does not play dictator game first.

The reader could remark that it seems possible to obtain the same results in the standard probabilistic representation through an appropriate modification of the probability space. The next section addresses this question. Basically the argument is that in applications to Social Sciences when the explanation of empirical data calls for assuming changes in preferences for no good (plausible) reason, it may be more appropriate to model preferences as being indeterminate. The HSM then allows to make predictions that can be empirically tested. Two experiments are briefly sketched.

3.2 HSM and the classic probabilistic model

This section is intended to clarify the relationship between the proposed Hilbert space representation and the classical probabilistic representation.

Our typical setup is that of two observables, A with eigenvalues $\{1_A, 2_A, \dots, n_A\}$, and B with eigenvalues $\{1_B, 2_B, \dots, m_B\}$. So to treat both observables we express the Hilbert space \mathcal{H} as a direct sum in two different ways:

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, \dots, \oplus \mathcal{H}_{n_A} \text{ and } \mathcal{H} = \mathcal{H}_{1_B} \oplus, \dots, \oplus \mathcal{H}_{m_B}$$

where \mathcal{H}_{i_A} is the subspace of eigenvectors of A with eigenvalue i_A (similarly

of \mathcal{H}_{j_B}). Given the state $|\psi\rangle \in \mathcal{H}$ of the agent under consideration, it can be represented as

$$|\psi\rangle = \sum_i \lambda_i |i_{A\psi}\rangle, \quad \sum_i |\lambda_i|^2 = 1, \quad |i_{A\psi}\rangle \in \mathcal{H}_{i_A}, \quad i = 1, \dots, n \quad (7)$$

Similarly,

$$|\psi\rangle = \sum_j \nu_j |j_{B\psi}\rangle, \quad \sum_j |\nu_j|^2 = 1 \quad |j_{B\psi}\rangle \in \mathcal{H}_{j_B}, \quad j = 1, \dots, m \quad (8)$$

Now, states $|j_{B\psi}\rangle \in \mathcal{H}_{j_B}$ of eq. (8) can also be written as a superposition of the form (7)

$$|j_{B\psi}\rangle = \sum_i \mu_{ij} |i_{Aj_B}\rangle, \quad \sum_i |\mu_{ij}|^2 = 1 \quad |i_{Aj_B}\rangle \in \mathcal{H}_{i_A}, \quad i = 1, \dots, n \quad (9)$$

Note that the vectors $|i_{A\psi}\rangle$ and $|i_{Aj_B}\rangle$, being both in \mathcal{H}_{i_A} , are eigenvector of A with the same eigenvalue i_A . The *non commutativity* of A and B is the violation of the equation

$$p_A(i_A) = p_{AB}(i_A), \quad i = 1, \dots, n \quad (10)$$

which is $|\lambda_i|^2 = \sum_j |\mu_{ij}|^2 |\nu_j|^2$. That is, the probability distribution on the outcomes of the measurement A may be affected by the fact that we measured B first and then A .

Can this be modelled in a standard probability theory? The answer is obviously yes. To do that let A be a random variable with values in $\{1_A, 2_A, \dots, n_A\}$ with probabilities $p_A(i_A) = \alpha_i \equiv |\lambda_i|^2$, $i = 1, \dots, n$ which will be the equivalent of eq. (7). Similarly the equivalent of eq. (8) is obtained if B is a random variable with values in $\{1_B, 2_B, \dots, m_B\}$ and with probabilities $p_B(j_B) = \beta_j \equiv |\nu_j|^2$, $j = 1, \dots, m$.

What is the analogue of eq. (9)? For each value j_B of B there is a probability distribution p_{Aj_B} of the random variable A given by

$$p_{Aj_B}(i_A) = \gamma_{ij} \equiv |\mu_{ij}|^2; \quad i = 1, \dots, n, \quad j = 1, \dots, m \quad (11)$$

Then, when measuring B and then A , the distribution of outcomes is given by

$$p_{AB}(i_A) = \sum_j \beta_j \gamma_{ij} \quad i = 1, \dots, n.$$

Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_m)$ and $\gamma = (\gamma_{ij})_{i=1, \dots, n, j=1, \dots, m}$, then the analogue of eq. (10) is

$$\alpha = \beta\gamma \quad (12)$$

which is generally not satisfied unless we impose restriction on α, β and γ .¹³

The following facts are easily verified :

- The equality (12) holds if and only if there is a probability distribution p on $\{(i_A, j_B); i = 1, \dots, n, j = 1, \dots, m\}$ whose marginal distribution of A and B are p_A and p_B respectively.

In such a case $p_{A|j_B}$ is just the conditional probability

$$p_{A|j_B}(i_A) = p(i_A | j_B) \quad (13)$$

and eq. (12) is then just the fundamental equation of conditional probability namely, for $i = 1, \dots, n$

$$p(i_A) = \sum_j p(j_B)p(i_A | j_B)$$

- In the HSM framework the eq. (10) holds if and only if the observables A and B commute, we conclude that
 1. Two commuting decision situations A and B can be modelled by *one* standard probability space whose elementary events are the pairs (i_A, j_B) of values of the two observables.
 2. Two non commuting DSs A and B cannot be modelled by a single probability space. In particular $p_{A|j_B}(i_A)$ *is not a conditional probability* in the formal sense since there is no underlying probability space in which the probabilities $p(i_A, j_B)$ is defined.

Of course not all instances of non-commutativity in decision theory call for Hilbert space modelling. In particular, the standard modelling approach is satisfactory when the modification of the probability space following the first decision, can be expected. As an example of this kind, consider the

¹³As an example take $\alpha = \beta = (\frac{1}{2}, \frac{1}{2})$ and $\gamma = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix}$ then $\gamma\beta = (\frac{3}{8}, \frac{5}{8})$.

following two decision situations: A is a plan for the evening which involves a choice between going to a (long) opera play (1_A) or staying at home (2_A). The decision situation B is an afternoon party where the agent is invited to choose between drinking whisky (choice 1_B) and drinking champagne (choice 2_B). Assume that the probability for the direct choice are the following:

$$p_A(1_A) = 0.3, p_A(2_A) = 0.7 \text{ while } p_B(1_B) = .5 = p_B(2_B).$$

Let us further assume that we performed an experiment and observed frequencies of choice corresponding to the following probabilities: $p_A(1_A|1_B) = 0.1$ and $p_A(1_A|2_B) = 0.15$. It is easy to verify that equation (10) is not satisfied. Indeed in this example there is no single probability model for which the probability $p(1_A, 1_B)$ can be defined. In such a case we clearly see a good reason why playing B before A modifies the probability distribution of the choices in A .

The example above hardly qualifies as a behavioral anomaly because we *do* expect preferences to be affected by the consumption of alcohol. Our point is that many instances of behavioral anomalies are very different from the example above. Those are instances in which we expect consistency with standard probabilistic model - because nothing justifies a modification of preferences - yet, actual behavior *is* ‘inconsistent’. *The Hilbert space model of preferences is useful when i) the conditional probability formula doesn’t hold and ii) there is no good reason for why playing the intermediary DS would modify the probability distribution for the final DS*. In such cases, a model in which the initial state (preference) is indeterminate may become more appealing.

Situations characterized by i) and ii) are routinely encountered in experimental psychology or in preferences test. One example is provided by Erev, Bornstein and Wallsten (1993). They describe an experiment showing that revealed preferences are not consistent with the so called *Information Reduction assumption* (IR) which assumes that uncertainty can be reduced to a numerical probability. This assumption is implicitly made by most theorists including e.g. Savage (1954) but also by many decision theorists including Kahneman and Tversky (1979). One implication of IR is that subjects should make the same decisions regardless of whether they are asked to state their probabilities or not. In an experiment EBW (1993) compare the choices made by two groups one of which is invited to answer a questionnaire before playing a simple version of a n-person game called the *intergroup public good*.¹⁴

¹⁴The IPG game proceeds as follows. Each subject in the experiment is given a sum of

EBW found that asking subjects to fill in a questionnaire stating their subjective probabilities for others' play *did* affect their subsequent choice. The revealed preferences in the two treatments were significantly different: In the group that played directly there was a positive correlation between the size of the expected price and the choice of investment (risky choice). While in the group that answered the questionnaire no such correlation existed. The authors propose that decision-makers asked to state probabilities or to give numerical assessments, pay less attention to the payoff dimension. We note that this interpretation is consistent with the assumption that the subjects are initially in a state of 'hesitation'. The questionnaire operates on this state of mind, triggers a determination along a dimension correlated with the choices and therefore affects the distribution over final choices. This example may also be explained, using the HSM formalism, in terms of framing effect see section 3.3.

We argue that if preferences change so 'easily' (for instance under the sole influence of filling a questionnaire), a more attractive approach may be to view the agent as being in an indeterminate state. The Hilbert space formalism models this indeterminacy of preferences and provides a general framework to make predictions in this kind of situations. It allows for a unified treatment of situations consistent with stable innate preferences (when two observables can be modelled by a single probability space), and situations inconsistent with stable innate preferences. Most of the psychological explanations given for anomalies such as extremeness aversion, anchoring framing etc.. seem to be in line with our main idea that preferences are more appropriately modelled as being indeterminate and context dependent.

3.2.1 Experiment 1: superposition of states of mind

Consider two identical populations of agents. Suppose that we can distinguish between two types of agents: altruist (a) or selfish (s). Assume that the type of the agent can be revealed by making him answer to a questionnaire. We propose an experiment where the two populations are invited to play a Prisoner Dilemma against a hidden player. Before playing the PD the agents from the first population are invited to answer the questionnaire. The agents

money, 5 Shekel, and chooses whether to keep it and cash in at the end of the session or to invest it. Members of the group with the higher number of investors receive a monetary reward of r Shekel each (In the experiment r was either 15 or 25 Shekels). In case of tie, members of both groups received $r/2$ units each.

from the second population play the PD game without any ‘preparation’.

Assume that the experiment is made on the first population of agents and shows that the population consists of a proportion of α of altruist agents (and a proportion $(1 - \alpha)$ of selfish agents). Denote by B the operator representing the questionnaire. Now, the agents are invited to play the Prisoner’s Dilemma game (operator A) with the same payoff matrix as in the example above:

	$[C]$	$[D]$
$1_A(C)$	3 3	4 0
$2_A(D)$	0 4	1 1

Assume that we find that a proportion β of altruist agents choose to cooperate ($(1 - \beta)$ choose to defect) and a proportion γ of selfish agents choose to cooperate ($(1 - \gamma)$ to defect). The proportion of agents of the first population choosing C is thus

$$p_1(C) = \alpha\beta + (1 - \alpha)\gamma.$$

The second part of the experiment is made on the other population of agents who play the PD game directly. From a classical point of view, agents are either altruist or selfish independently of the fact that they have answered the questionnaire (the types are fixed). Since the two populations are identical, we should be able to apply the conditional probability formula (although the magnitudes in the expression on the rhs of the equation below have not been measured for population 2) to obtain

$$p_2(C) = p(a)p(C|a) + p(s)p(C|s) = \alpha\beta + (1 - \alpha)\gamma.$$

Assume that we find that the proportion of agents in population 2 who choose C is $p'(C)$ which is *significantly different* from $p_2(C)$ above. The only way to explain such results within a classical framework would be to give up the conditional probability formula and hence to assume that the agents, though being all the time in a definite state (altruist or selfish) change the way they play the Prisoner’s Dilemma game by the sole virtue of having answered the questionnaire. This assumption does not seem to be consistent with the idea of an agent having fixed determinate preferences.

We propose that at the beginning, the agent is in a superposition of the agent is a superposition of two types: altruist and selfish. Both modes of

behavior are 'latent' and have the potential to be observed. As the agent answers the questionnaire, this 'hesitation' is resolved: he 'has' to choose between the two behavioral inclinations (propensities to act). Before this operation, the two modes of behavior affect his choice (in the PD) but as soon as one mode has manifested itself, only this mode is active. If in fact the results with respect to the choice in the Prisoner's Dilemma turn out to be different in the two situations (having answered the questionnaire or not), the proposed experiment would provide support for the hypothesis that preferences are indeterminate i.e. that agents can be represented by a superposition of types.

3.2.2 Experiment 2: Order matters

Consider three different DSs each with a choice between two alternatives: PD (Prisoner's Dilemma), DG (Dictator game) and UG where UG denotes the responder's decision situation in an ultimatum game where the first mover (a hidden player) proposes a given split of a known sum of money. The decision is whether to accept or to reject the split. In case of a rejection, the payoff is zero to both players, in particular to our decision maker - the responder. Suppose that we have empirically established that order matters for the following two pairs of DSs (PD, DG) and (DG,UG). According to our theory, these DSs must be represented by pairs of non-commuting operators. Then, unless the behavior types in the PD and UG are perfectly correlated i.e. that the proportion of players having chosen 'cooperate' in the PD and choosing 'accept' in the UG is either equal to 1 or to zero, our theory predicts that order matters in the play of the pair of DSs (PD,UG).¹⁵

In order to test this prediction we propose the following experiment. We first establish whether or not order matters between the two first pairs of DSs (PD,DG) and (DG,UG) using the following procedure: We take two populations of *identical* individuals (assumed to be represented by the same state vector). The first population is invited to play the following sequence: first the PD (against a hidden player as in experiment 1) and thereafter

¹⁵Technically speaking given three operators A, B, C (with two eigenvectors each), if both (A, B) and (B, C) are pairs of non-commuting operators, then the eigenvectors of A as well as those of C form a basis of the same two dimensional space. If in addition A and C commute then A and C must have the same eigenvectors. The two operators differ only by the names given to the eigenvalues. In other words the types in A are perfectly correlated with the types in C .

the DG (against a hidden third player). The individuals from the second population are invited to play the same DSs but in the opposite order: first the DG and thereafter the PD. If the distributions of choices turn out to be significantly different we have established that order matters when playing the PD and DG. We perform a similar test for (DG,UG) (with another set of participants). If we do observe that order matters even in this case, we move over to the next step.

The next step of the experiment is to test whether or not the types in PD and UG are perfectly correlated and we do this by selecting a population of agents who have chosen ‘cooperate’ in the PD. Assume that there is no evidence for perfect correlation we can then move to the last phase of the experiment namely testing the prediction that PD and UG are non-commuting decision situations. We do that by the same procedure we used to test the non commutativity of (PD,DG) and (DG,UG) (of course with another set of participants). If we do in fact observe that the distributions of choices are significantly different it will be consistent with our theory of preferences indeterminacy.

Our objective is to take two examples of experiments where two different behavioral theories have been proposed e.g. the ‘transfer effect’ for the WL and the PD games. Our contribution would be to link the two experimental phenomena with a ‘cross’ prediction along the lines described above. If the prediction turns out to be correct but cannot be explained by either one of those theories, we would have demonstrated that our theory has the potential to both unify some of the existing body of behavioral economics and generate new results.

3.3 Framing Effects

Providing a framework to think about framing effects is one of the interesting and promising application of the indeterminacy approach in decision theory. Kahneman and Tversky (2002 p. xiv) define the framing effect using a two steps (non formal) model of the process of decision-making. The first step corresponds to the construction of a representation of the decision situation. The second to the evaluation of the choice alternatives. The crucial point is that *“the true objects of evaluation are neither objects in the real world or verbal description of those objects; they are mental representations.”* To capture this feature we propose that decision-making be modelled as a sequence of operators i.e. as a process whereby the agent is successively subjected

to two operators. Thus, we suggest to model frames as operators in a way similar to decision situations. A frame or FR is defined as a collection of alternative mental representations (of a decision situation). The corresponding operator is denoted \tilde{A} where the tilde is used to distinguish operators associated with FRs from those associated with DSs. We model the process of constructing (selecting) a representation as the analogue of the 'measurement' of the operator corresponding to the framing proposed to the agent i.e. the collapse onto one of the frame operators' eigenvectors (onto one of the alternative mental representations).¹⁶ In contrast with types associated with a DS, the types associated with a FR (hereafter we call them FR-type) are not directly observable but they could, in principle, be revealed by a questionnaire.

The operators associated with FRs and DSs are defined within the same state space of representations/preferences. One interpretation is that representations and preferences have grown entangled through the activity of the mind i.e. decision processes.¹⁷ We provide now an illustration of this approach.

In Selten (1998) an experiment performed by Pruitt (1970) is discussed. Two groups of agents are invited to play a Prisoner Dilemma. The first group is presented the game in the usual matrix form but with the choices labelled 1 and 2 (instead of C and D presumably to avoid associations with the suggestive terms 'cooperate' and 'defect'):

	$[C]$	$[D]$
$1_G(C)$	3 3	4 0
$2_G(D)$	0 4	1 1

The second group is presented the game with payoffs decomposed as

¹⁶Interestingly, this approach is consistent with research in neurobiology that suggests that the perception of a situation involves operations in the brain very similar to those involved when choosing an action. 'To perceive is not only to combine, weight, it is to select'. (p.10 Berthoz 2003)

¹⁷In quantum physics the entanglement of the states of two spin 1/2 particles in the singlet spin state, means that the two particles form a single system that cannot be factored out into two subsystems even when separated by a great distance (cf. the famous Einstein Podolsky Rosen (EPR) non-locality paradox).

follows:

	For me	For him
1_G	0	3
2_G	1	0

The payoffs are the sums of what you take for yourself and what you get from the other player. Game theoretically it should make no difference whether the game is presented in the matrix form or in the decomposed form. However, an experiment made by Pruitt shows that one observes dramatically more cooperation in the decomposed form presentation.

Selten proposes a ‘bounded rationality’ explanation: the players would be making a superficial analysis and do not perceive the identity of the game presented in the two forms. We propose that the agents after having been confronted with the matrix presentation are not the same as before and not the same as the agents having been confronted with the decomposed form. The object of evaluation does not pre-exist the interaction between the agent and the framed decision situation. It is, as Kahneman and Tversky propose, constructed in the process of decision-making.

Using the framework developed in this paper, we call \tilde{A} the FR operator corresponding to the framing of the matrix presentation, \tilde{B} the one corresponding to the framing of the decomposed presentation and G the DS operator associated with the game. G has two non-degenerated eigenvalues denoted 1_G and 2_G . The first step in the decision process corresponds to the selection of a mental representation of the decision situation. At that stage the agent is subjected to a frame \tilde{A} or \tilde{B} . Accordingly, before being presented the DS in some form, the agent is described by a superposition of possible mental representations. For the sake of illustration we assume that each frame only consists of two alternative representations. We can write $|\psi\rangle = \alpha_1 |\tilde{a}_1\rangle + \alpha_2 |\tilde{a}_2\rangle$ and $|\psi\rangle = \beta_1 |\tilde{b}_1\rangle + \beta_2 |\tilde{b}_2\rangle$. We propose the following description of the mental representations (this is only meant as a suggestive illustration)

- $|\tilde{a}_1\rangle$: decision situation perceived as an artificial small stake DS;
- $|\tilde{a}_2\rangle$: decision situation perceived as a real stake DS by analogy with real life PD like situations (often occurring in a repeated setting).
- $|\tilde{b}_1\rangle$: decision situation perceived as a test of generosity;
- $|\tilde{b}_2\rangle$: decision situation perceived as a test of smartness.

Given any initial state, subjecting the agent to either frame leads to a collapse on either one of the two mental representations associated with that frame. So the state of the agent becomes some $|\tilde{a}_i\rangle$ or $|\tilde{b}_j\rangle$ ($i, j = 1, 2$) expressed in the basis of the decision situation, the state of our agent writes

$$\begin{aligned} |\tilde{a}_i\rangle &= \gamma_{1i}|1_G\rangle + \gamma_{2i}|2_G\rangle; \\ |\tilde{b}_j\rangle &= \delta_{1j}|1_G\rangle + \delta_{2j}|2_G\rangle. \end{aligned}$$

We can now express the framing effect as follows:

$$p_{G\tilde{A}}(1_G) \neq p_{G\tilde{B}}(1_G) \quad (14)$$

Using our result in section 2.3.2 we know that

$$\begin{aligned} p_{G\tilde{A}}(1_G) &= p_G(1_G) - I(\tilde{A}) \\ p_{G\tilde{B}}(1_G) &= p_G(1_G) - I(\tilde{B}) \end{aligned}$$

where $p_G(1_G)$ is the probability for choosing 1 in a (hypothetical) unframed situation, $I(\tilde{A})$ captures interference between the two FR-types \tilde{a}_1 \tilde{a}_2 and similarly for $I(\tilde{B})$. Hence, we have a framing effect whenever $I(\tilde{A}) \neq I(\tilde{B})$.

In our example $I(\tilde{A}) = 2 \operatorname{Re} \{\alpha_1 \gamma_{11} \alpha_2^* \gamma_{12}^*\}$ and $I(\tilde{B}) = 2 \operatorname{Re} \{\beta_1 \delta_{11} \beta_2^* \delta_{12}^*\}$. The experimental results discussed in Selten (1998) obtain with $I(\tilde{B}) < 0$ and $I(\tilde{A}) > 0$. In general, unless the FR-types belonging to \tilde{A} are perfectly correlated with those belonging to \tilde{B} or the FR operators both commute with the decision operator (in which case $I(\tilde{A}) = I(\tilde{B}) = 0$), we have $I(\tilde{A}) \neq I(\tilde{B})$ implying $p_{G\tilde{A}}(1_A) \neq p_{G\tilde{B}}(1_A)$. We thus propose that the framing effect arises as a consequence of (initial) indeterminacy of the agent's representation of the decision situation.

So far this is only a descriptive analysis, in order for it to gain some predictive power we would like to be able to identify properties of frame operators which are independent of the DS that follows or at least applicable to a class of DSs. One example of such a property could be that the choice situation is mentally represented as a test of personal qualities (cf \tilde{B} above).

We could then make predictions and test them in experiments. Another possible approach consists in focusing on the mental representation stage. We could experimentally investigate the effect on choice of a sequence of frame operators in different orders. We could also try to devise questionnaire that directly addresses the mental representation.

4 Concluding remarks

In this paper we propose an approach to decision-making that has the potential to provide a unified framework for explaining a variety of so called ‘behavioral anomalies’. The basic idea is that preferences, besides being typically unknown (to other agents) are also indeterminate. A main implication of indeterminacy of preferences is that a choice situation does not merely reveal the (fixed) preferences of an agent, but also contributes to constructing them. Consequently, in order to be able to make predictions one must account also for the context in which preferences are expressed. The Hilbert space model we propose provides a mathematical model for making such contextual predictions.

We argue that this abstract model may be an appropriate extension of Harsanyi’s model of uncertainty which accommodates empirically observed phenomena like context dependency of preferences or framing effects.

This explorative note is a first step of a larger research program intended to investigate the implications of the indeterminacy hypothesis in game theory and economic models. Those implications will then be tested experimentally.

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5 Appendix: Elements of Quantum Physics

5.1 States and Observables

In Quantum Mechanics the state of a system is represented by a vector $|\psi\rangle$ in a Hilbert space \mathcal{H} . According to the superposition principle, every complex linear combination of state vectors is a state vector. To each physical property of the system is associated a Hermitian operator called an observable.

Theorem 1

A Hermitian operator A has the following properties:

- its eigenvalues are real
- two eigenvectors corresponding to different eigenvalues are orthogonal
- There is an orthonormal basis of the relevant Hilbert space formed with the eigenvectors of A

Let us call $|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle$ the normalized eigenvectors of A forming a basis of \mathcal{H} . They are associated with eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$, so:

$A|v_i\rangle = \alpha_i|v_i\rangle$. The eigenvalues can possibly be degenerated, i.e. for some i and j , $\alpha_i = \alpha_j$. That means that there are more than one linearly independent eigenvectors associated with the same eigenvalue. The number of them is the degree of degeneracy of the eigenvalue. The dimension of the eigen subspace spanned by these eigenvectors is the degree of degeneracy of the eigenvalue. In this case, the orthonormal basis of \mathcal{H} is not unique because it is possible to replace the eigenvectors associated to the same eigenvalue by any complex linear combination of them to get another orthonormal basis. When an observable A has no degenerated eigenvalue there is a unique orthonormal basis of \mathcal{H} formed with its eigenvectors. In this case, (see below) it is by itself a Complete Set of Commuting Observables.

Theorem 2

If two observables A and B commute there is an orthonormal basis of \mathcal{H} formed with eigenvectors common to A and B .

Let A be an observable with at least one degenerated eigenvalue and B another observable commuting with A . There is no unique orthonormal basis formed with A eigenvectors. But there is an orthonormal basis of the relevant Hilbert space formed with eigenvectors common to A and B . By definition, $\{A, B\}$ is a Complete Set of Commuting Observables (CSCO) if this basis

is unique. Generally, a set of observables $\{A, B, \dots\}$ is said to be a CSCO if there is a unique orthonormal basis formed with eigenvectors common to all the observables of the set.

5.2 The measurement

To each physical property of a system S is associated an observable A . Let $|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle$ be the normalized eigenvectors of A associated respectively with eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ and forming a basis of the relevant state space. Assume the system is in the normalized state $|\psi\rangle$. A measurement of A on S obeys the following rules, called ‘Wave Packet Reduction Principle’ (in short Reduction Principle)

Reduction Principle

1. When a measurement of the physical property associated with observable A is done on a system S in a state $|\psi\rangle$, the result can only be one of the eigenvalue of A .

2. The probability to get the non degenerated value α_i is $P(\alpha_i) = |\langle v_i | \psi \rangle|^2$

3. If the eigenvalue is degenerated then the probability is the sum over the eigenvectors associated with this eigenvalue: $P(\alpha_i) = \sum |\langle v_i^j | \psi \rangle|^2$

4. If the measurement of A on a system S in the state $|\psi\rangle$ has given the result α_i then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigen subspace of the relevant Hilbert space associated with α_i . If the eigen value is not degenerated then the state of the system after the measurement is the normalized eigenvector associated with the eigenvalue.

If two observables A and B commute then it is possible to measure both simultaneously: the measurement of A is not altered by the measurement of B . That means that measuring B after having measured A does not change the value obtained for A . If we measure again A after the B measure we get again the same value for A . Both observables can have a definite value.

5.2.1 Interferences.

The archetypal example of interferences in quantum mechanics is given by the famous two slits experiment.¹⁸ A parallel beam of photons falls on a

¹⁸See e.g. Feynman (1980) for a very clear presentation.

diaphragm with two parallel slits and arrives on a photographic plate. A typical interference pattern showing alternate bright and dark rays can be seen. If one slit is shut then the previous figure becomes a bright line in front of the open slit. This is perfectly understandable if we consider photons as waves, as it is supposed in classical electromagnetism. The explanation stands on the fact that when both slits are open, one part of the beam goes through one slit and the other part through the other slit. Then, when the two beams join on the plate, they interfere constructively (giving bright rays) or destructively (giving dark ones) depending on the difference of length of the path they have followed. But a difficulty arises if photons are considered as particles as can be the case in quantum mechanics. Indeed, it is possible to decrease the intensity of the beam so as to get only one photon travelling at a time. In this case, if we observe the slits to detect when a photon pass through (for example by installing a photodetector in front of the slits), it is possible to see that each photon goes through only one slit. It's never the case that a photon splits to go through both slits. The photons behave like particles. Actually the same experiment was done with electrons instead of photons with the same result. If we do the experiment this way with electrons (observing through which slit the electrons go i.e. sending light on each slit to "see" the electrons), we see that each electron goes through one unique slit and in this case, we get no interference. If we repeat the same experiment without observing through which slit pass the electrons then we recover the interference pattern. Thus the simple fact that we observe which slit the electron goes through destroys the interference pattern (two single slit patterns are observed). The quantum explanation stands on the assumption that when we don't observe through which slit the electron has gone then its state is a superposition of both states "gone through slit 1" and "gone through slit 2"¹⁹ while when we observe it, its state collapses onto one of these states. In the first case, the position measurement is made on electron in the superposed state and gives an interference pattern since both states manifest in the measurement. In the second case, the position is measured on electrons in a definite state and no interference arises. In other words, when only slit 1 is open we get a spectrum, say S_1 (and S_2 when only slit 2 is open). We expect that when both slits are open we get a spectrum say S_{12} , sum of the two previous spectra, but this is not the case: $S_{12} \neq S_1 + S_2$.

¹⁹This doesn't mean that the photon actually went through both slits. This state simply can't be interpreted from a classical point of view.