

A DISCRETE-TIME STOCHASTIC MODEL OF JOB MATCHING

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Abstract

In this paper, an explicit micro scenario is developed which yields a well-defined aggregate job matching function. In particular, a stochastic model of job-matching behavior is constructed in which the system steady state is shown to be approximated by an exponential-type matching function, as the population becomes large. This steady-state approximation is first derived for fixed levels of search intensities, where it is shown (without using a free-entry condition) that there exists a unique equilibrium. It is then shown that if job searchers are allowed to choose their search intensities optimally, this model is again consistent with a unique steady state.

Keywords: discrete-time matching function, large population approximation, optimal search intensity.

JEL Classification: D83, J41, J61.

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1. Introduction

In the last few years, the vision of the labor market has changed from a stock perspective to a flow perspective. Indeed, labor markets in both the United States and Europe are characterized by very important gross flows that are associated with high rates of job creation and job destruction (Blanchard and Diamond, 1989, Burda and Wyplosz, 1994, Davis, Haltiwanger and Schuh, 1998). Because of these large flows, the labor market cannot perfectly match workers and jobs, so that vacant jobs and unemployed workers coexist.

Mortensen and Pissarides were the first to develop a unified theoretical framework for analyzing this complex matching process, involving mutual search by workers and firms (for a synthesis, see Pissarides, 1990, and Mortensen and Pissarides, 1999). This has become widely known as the Mortensen-Pissarides model (MP hereafter). A central feature of their analysis is the aggregate matching function which combines job vacancies and unemployed workers to yield new active jobs. As pointed out by Pissarides (1990) and Blanchard and Diamond (1989), this notion of a matching function hides a complex reality in which skill differences between workers and jobs have a major role to play.

In the present paper, there is assumed to exist a fixed population of workers with heterogeneous skills. Firms are similarly characterized by a flow of heterogeneous production possibilities, with newly profitable jobs being opened and unprofitable jobs being closed. Thus job matching is taken to constitute a process whereby heterogeneous workers compete for jobs with different skill requirements. Here heterogeneity of workers need not imply any superiority or inferiority among their abilities. Rather, all are assumed to possess the same level of general human capital, which is manifested in a variety of different skills (as for example college graduates with degrees in different fields).¹

In this context, our first objective is to derive an explicit micro scenario that leads to a well-defined job matching function. This scenario involves a day-to-day process in which currently vacant jobs are posted by firms, and currently active job seekers apply for these vacancies. A vacancy is filled on a given day if and only if from among all those currently active job seekers, at least one applies for the position who meets all the job requirements. From these conditions one may derive an explicit job-filling probability on any given day.² The corresponding daily matching function is then given by the product of this probability and the expected number of vacant jobs per worker on each day. When the population of workers

¹This approach to modeling heterogeneity on both sides of the labor market is similar to that of Hamilton, Thisse and Zenou (2000) and to Sattinger (1993), although they focus on different issues since, in particular, there is no search in their analysis.

²Equivalently, one may also determine the job-hiring probability that a job seeker is employed on a given day.

is allowed to become large, it is shown (Section 2) that the asymptotic form of this matching function is of exponential type and has the standard properties, i.e., is concave, monotone increasing in both arguments, and homogeneous of degree one. However, a comparison of this matching function with its finite-population counterpart reveals that while the latter is also increasing and concave, it fails to be linearly homogeneous. Hence in the present case, linear homogeneity turns out to be more a property of limits than of the underlying behavioral process.

This exponential-type matching function (which is essentially a limiting form of the classical 'urn model' in discrete probability theory) was first employed in a market context by Butters (1977) to model contacts between buyers and sellers in commodity markets. In the labor literature, a common story is that workers know where vacancies are, but do not know which particular vacancies other workers will visit, allowing for the possibility that some workers are unable to fill vacancies because they were 'second in line'. This structure reduces the aggregate meeting process to an 'urn-ball' process in which the labor market is visualized as m 'urns' (i.e. vacancies) and n 'balls' (i.e. workers), each ball having a probability $1/m$ of being directed to any given urn. Hall (1979a,b), Pissarides (1979), Peters (1991), Blanchard and Diamond (1994) and Burdett, Shi and Wright (2000), all derive the number of contacts that will take place in some interval, as a function of the number of vacancies and searching workers that is immediately implied by this structure.

It should however be clear that our derivation mainly differs from the usual treatment by these authors in that we carefully set up the finite, stochastic laws of motion for the stock of vacancies and unemployed workers (and then take the appropriate limits in steady state, as the population size gets large), instead of just starting off with the corresponding deterministic approximations that obtain for large numbers of agents. Another extension of our paper is the assumption that workers are explicitly heterogeneous so that some of them do not have the skill needed for the jobs that they contact. This introduces a parameter for 'skill mismatch' into the framework, which is usually introduced in an ad hoc fashion in the empirical literature.

Given this micro-based matching function, we begin with a simple stochastic model of the labor market (section 2) in which search intensities are assumed to be given. Here the focus is primarily on the steady-state behavior arising from the interaction of a finite population of workers with a flow of job opportunities linked by this matching function. This model departs from the MP model in terms of the basic flow of job creations and destructions. Such flows are here modeled as a type of stochastic birth-death process in which profitable jobs are 'born' and unprofitable jobs 'die' at rates depending on the overall state of the economy. In Blanchard and Diamond (1994) (BD hereafter), it is assumed rather

that there exists a fixed stock of potentially productive jobs which continually switch between 'productive' and 'unproductive' states, depending on the economy. A steady state in both these models then involves a balancing of this job flow with the alternating flow of workers between employment and unemployment. In contrast, such a balance is achieved in MP by assuming a 'free entry' condition in which firms continue to create new job vacancies until the expected gains from advertising new jobs fall to zero. But while this condition is quite natural within a given industry where jobs are similar in nature, it is more difficult to interpret in a stochastic environment where individual job creations and closings are treated as essentially independent events.³

As in both the MP and BD models, we then show that when the population of workers becomes large, there exists a unique steady-state for this labor market characterized by the mean job-vacancy rate and unemployment rate.⁴ In particular, the steady-state unemployment rate is shown to be positive. Hence the imperfect nature of this job-matching process always gives rise to permanent frictional unemployment regardless of how many jobs are available.

Given this basic steady-state model, our second objective is to relax the assumption of fixed search intensities, and to derive an explicit optimal search intensity level for unemployed workers. In particular, we focus on the utility-maximizing problem of an unemployed worker in deciding how much time to spend in job search (i.e., how many days per week to actively search for a job). Here it is shown (section 3) that the optimal strategy is to search up to the point where the marginal gain of additional search (in terms of decreased unemployment duration) equals its marginal cost (in terms of reduced leisure). We then show that this behavior is consistent with the steady-state model above in the sense that there now exists a unique steady state equilibrium with endogenous search.

³In particular, if jobs are similar in nature, then changes in the economy will tend to affect all jobs alike. Thus, in this context, it is more difficult to support the usual assumption that individual job openings and closings are independent 'Poisson' events.

⁴In both the MP and BD models, this argument is implicitly made by passing directly to the limit. For example, the BD model involves a system of deterministic difference equations which represent the transitions in mean unemployment and job vacancy levels from week to week. Here we begin with the actual stochastic steady states for each finite population size, and show that (under mild conditions) the corresponding steady-state unemployment and job-vacancy rates converge in probability to their mean values as the population size becomes large. This provides an explicit probabilistic foundation for the resulting steady-state equations in terms of mean unemployment and job-vacancy rates.

2. The Basic Model

Consider a population of N workers who compete for jobs in a given labor market. All jobs are offered at the same prevailing daily wage, w , but are assumed to be completely specialized in terms of skill requirements.⁵ Similarly, workers are assumed to be heterogeneous in terms of their skill endowments. As stated in the introduction, job matching in the present context constitutes a process whereby heterogeneous workers allocate themselves to jobs with different skill requirements. Heterogeneity of workers does not here imply any superiority or inferiority among their abilities. Rather, all are assumed to possess the same level of general human capital, which is manifested in a variety of different skills. Hence all workers are assumed to have the same chance of being qualified for any given job, as modeled by a common qualification probability; θ .

In this context, the actual job matching process can be described as follows. At any point in time, each worker is either employed or unemployed, and only unemployed workers are assumed to search for jobs. Since individual jobs are completely specialized, their creation and closing can be regarded as independent events. In particular, job creations and job closings are here modeled as a simple birth-death process in which 'births' are governed by a job-creation rate, λ (denoting the mean number of jobs per worker created each day) and 'deaths' are governed by a job-closure rate, μ (denoting the probability that any currently existing job will be closed on a given day). This process is taken to depend on the general state of economy, and hence is treated as exogenous to the labor market. Two basic parameters in the model are initially taken to be given: the daily wage, w ; and the search intensity, s ; which denotes the expected fraction of days on which each unemployed worker actively seeks a job. The following is a brief description of the behavioral day-to-day scenario for the job market model on a given day, t :

- ² At the beginning of day t those unemployed workers currently seeking work appear at the job market. All current job vacancies are posted, and are offered at the going wage w . Each searcher applies for a single job. No additional prior information about jobs is available, and there is no communication between searchers. Hence searchers choose jobs at random, and more than one searcher may apply for the same job.
- ² As mentioned above, each job applicant has the same probability, θ , of satisfying all qualifications for the given job. If more than one applicant is

⁵The assumption of completely specialized jobs can also be found in Pissarides (1990) and Blanchard and Diamond (1994).

qualified for a job, the employer chooses a qualified applicant at random. Otherwise the job is not filled on day t .

- 2 At the end of day t each successful applicant is notified, and is requested to start work on the following day. In addition, decisions are made by employers as to which jobs are no longer profitable and should be closed. For currently active jobs which are closed, layoff notices are distributed to workers. Moreover, for jobs which are filled that day and then closed, the successful (but unlucky) applicants are also given notices. Finally, those currently vacant jobs which are closed are simply removed from the postings at the beginning of the next day. As mentioned above, all jobs (active or vacant) have the same chance, $\frac{1}{2}$, of being closed on day t .
- 2 In addition, those new job opportunities which have arisen during the day (at rate, λ , per worker) are added to the vacant job postings for the next day.

This process is governed by the following system of accounting equations:

$$V_{t+1}^N = \overset{\text{Beginning of Day } t}{V_t^N} + \overset{\text{End of Day } t}{(B_t^N - F_t^N - C_{vt}^N)} \quad (2.1)$$

$$U_{t+1}^N = U_t^N + C_{at}^N - F_t^N \quad (2.2)$$

Here the random variables, V_t^N and U_t^N , denote respectively the numbers of job vacancies and unemployed workers at the beginning of day t . Variable F_t^N denotes the number of vacant jobs filled by the end of day t . Variables C_{vt}^N and C_{at}^N denote respectively the numbers of vacant jobs closed and active jobs closed at the end of day t (removed from the postings on day $t + 1$). Finally, variable B_t^N denotes the number of new job openings announced at the end of day t . (The reason for the population superscript N will become clear later). Equation (2.1) then states that the change in total vacant jobs from day t to $t + 1$ is given by the difference between the new jobs created and the vacant jobs either filled or closed on day t . Similarly, equation (2.2) states that the change in unemployment from day t to $t + 1$ is given by the difference between the number of workers laid off (i.e., the number of active jobs closed) and the number of workers hired (i.e., the number of vacant jobs filled) on day t .

2.1. Job Matching Process

Within this overall accounting framework, we now focus on the key variables, F_t^N , which summarize job hiring activity on each day. To do so, we begin by fixing the number of vacant jobs and unemployed workers, say $V_t^N = m$ and $U_t^N = n$ (where it is assumed that there is at least one vacant job, i.e. $m \geq 1$). In this context, if for each vacant job, $j = 1; \dots; m$, we define the indicator variable

$$F_t^j = \begin{cases} 1, & \text{if job } j \text{ is filled on day } t \\ 0, & \text{otherwise} \end{cases} \quad (2.3)$$

then by definition, the conditional value of F_t^N given m is of the form

$$F_t^N = \sum_{j=1}^m F_t^j : \quad (2.4)$$

Hence to model F_t^N we begin by considering the conditional distribution of each F_t^j given $V_t^N = m$ and $U_t^N = n$. To do so, recall first that not all unemployed workers necessarily seek work on any given day. Rather, there is a probability (search intensity), s , that any worker chooses to search on day t . Hence, assuming independent decision behavior by all individuals, the probability that l ($0 \leq l \leq n$) individuals apply for jobs on day t is given by the binomial probability:

$$P(l|n) = \frac{\bar{A}!}{l! (n-l)!} s^l (1-s)^{n-l} : \quad (2.5)$$

Next, recalling that jobs are chosen at random by applicants, the chance that any worker applies for job j is given by $1/m$. Hence the chance of k ($0 \leq k \leq l$) applicants for job j is given by

$$P_m(k|l) = \frac{\bar{A}!}{k! (l-k)!} \left(\frac{1}{m}\right)^k \left(1 - \frac{1}{m}\right)^{l-k} ; \quad (2.6)$$

A straightforward calculation then shows that the probability of k applicants given m and n is of the form:

$$P(k|m; n) = \frac{\bar{A}!}{k!} \frac{\bar{A}!}{s^k} \left(\frac{1}{m}\right)^k \left(1 - \frac{s}{m}\right)^{n-k} \quad (2.7)$$

Finally, recalling that workers are qualified for any given job with probability, ϕ , and that a job is filled only if at least one qualified worker applies, it follows that the probability of filling job j given k applicants is

$$P(F_t^j = 1|k) = 1 - (1 - \phi)^k : \quad (2.8)$$

Hence, combining (2.7) and (2.8), we easily obtain:

$$P^3 F_t^j = 1jm; n = 1 \sum_{k=0}^n (1-i)^k \frac{n!}{k!} \left(\frac{s}{m}\right)^k \mu \left(1-i\right)^{n-k} \frac{s^{n-k}}{m^{n-k}} : \quad (2.9)$$

But recalling that the moment generating function of the binomial, $B(p; n)$, is of the form, $E \mu^k = (1-i-p + \mu p)^n$, it follows that

$$P^3 F_t^j = 1jm; n = 1 \sum_{k=0}^n \mu^k \left(1-i\right)^{n-k} \frac{s^{n-k}}{m^{n-k}} : \quad (2.10)$$

Finally, observing that this probability is precisely the conditional mean, $E(F_t^j | jm; n)$, of the random variable F_t^j , it follows from (2.4) that the expected total number of jobs filled on any day for which there are m vacant jobs and n unemployed workers is given by

$$E(F_t^N | jm; n) = \sum_{j=1}^n E(F_t^j | jm; n) = m \sum_{j=1}^n \left(1-i\right)^{n-j} \frac{s^{n-j}}{m^{n-j}} \mu : \quad (2.11)$$

The function ϕ_N defined by the right hand side, i.e. by

$$\phi_N(m; n) = m \sum_{j=1}^n \left(1-i\right)^{n-j} \frac{s^{n-j}}{m^{n-j}} \mu : \quad (2.12)$$

is thus seen to summarize the key workings of this job market, and is designated as the matching function for the system.

By employing arguments similar to those above, one can in principle analyze the day-to-day stochastic behavior of this finite-population system. In particular, one may determine the explicit form of the discrete conditional distribution of F_t^N given $V_t^N = m; U_t^N = n$ ⁶. This, together with the properties of the (discrete-time) birth-death model for job creations and closings, allows one to determine the conditional distribution of $V_{t+1}^N; U_{t+1}^N$ given $V_t^N; U_t^N$ [by employing the basic accounting equations (2.1) and (2.2)]. For our present purposes, it suffices to observe that the Markov chain defined by these conditional distributions is necessarily both positive recurrent and acyclic, so that for each population size N this process converges to a unique steady-state distribution [as for example in Kulkarni (1995, Theorem 5.3.15)].

While the exact form of this steady-state distribution can be exceedingly complex for small N , it turns out that under fairly mild conditions, this distribution

⁶The case of $V_t^N = 0$ must be treated separately. By our assumptions, it follows that $P^1 F_t^N = 0 = C_{vt}^N V_t^N = 0 = 1$. Hence $P^1 V_{t+1}^N = m_j V_t^N = 0 = P^1 B_t^N = m_j V_t^N = 0 = P^1 B_t^N = m$ and $P^1 U_{t+1}^N = n_j V_t^N = 0; U_t^N + C_{at}^N = n = 1$.

concentrates in the neighborhood of its mean as N becomes large.⁷ Hence, following the standard approach adopted in most of the literature, we henceforth assume that the system is in steady state, and that N is sufficiently large to allow steady-state fluctuations in both unemployment levels and job vacancies per worker to be ignored. We now develop this nonstochastic approximation for large N .

2.2. Mean Steady-State Relations

If the system is in steady state, then the mean values of all random variables in (2.1) and (2.2) are constant over time. Hence, letting $V^N; U^N; F^N; C_a^N; C_v^N$, denote the steady-state counterparts of the random variables in these two equations, it then follows that $E V_t^N = E V^N$ for all t , and similarly for all other random variables. By replacing all random variables in (2.1) and (2.2) with their steady-state counterparts and taking expectations, we obtain the following (reduced) steady-state relations:

$$E F^N + E C_v^N = E B^N \quad (2.13)$$

$$E F^N = E C_a^N \quad (2.14)$$

Recall next from the basic model description that each of the $V^N; F^N$ vacant jobs remaining at the end of the day has the same chance, $\frac{1}{2}$, of being closed (removed from the job postings), so that the expected number $E C_v^N$ of vacant jobs closed is given by:

$$\begin{aligned} E C_v^N &= \sum E C_v^N j V^N; F^N = k P V^N; F^N = k \\ &= \sum (\frac{1}{2}k) P V^N; F^N = k \\ &= \frac{1}{2} E V^N; F^N = \frac{1}{2} E V^N; E F^N \end{aligned} \quad (2.15)$$

Similarly, each active job has the same chance, $\frac{1}{2}$, of being closed. But since active jobs include both those jobs active at the beginning of the day and those filled that day, the number of active jobs is identical to the number of currently employed workers, $N; U^N$, plus the number of vacancies filled, F^N . Hence it follows from the same argument as in (2.15) that the expected number, $E C_a^N$, of active jobs closed is given by:

$$E C_a^N = \frac{1}{2} N; E U^N + E F^N \quad (2.16)$$

⁷See for example the results of Brummelle and Gerchak (1980).

Finally, by substituting (2.15) and (2.16) into (2.13) and (2.14) and rearranging terms, we may conclude that in the steady state:

$$(1 - \beta) E^3 F^N + \beta E^3 V^N = E^3 B^N \quad (2.17)$$

$$(1 - \beta) E^3 F^N = \beta N_j E^3 U^N \quad (2.18)$$

2.3. Large Population Approximations

Given this reduced system of relations, we next ask how the steady-state distributions of B^N ; U^N ; V^N ; F^N behave as the population N becomes large. Each of these random variables will be considered in order.

2.3.1. Job Creation Rate.

It should be clear from the basic model described above that this economy is largely driven by the creation of new job opportunities. Moreover, it was implicitly assumed that the per-worker rate of job-creation, \bar{b} , is independent of population size. In other words, increments in population (and hence in potential demand for goods and services) are assumed to generate proportional increments in mean job creations. This can be modeled formally by assuming that for each individual i there is a random variable, B_i , representing the daily jobs created by the presence of i in the economy. This convention allows the total number of jobs created to be represented as a sum

$$B^N = \sum_{i=1}^N B_i \quad (2.19)$$

If it then assumed that these individual contributions are independently and identically distributed, with mean \bar{b} and finite variance σ_b^2 , then by the Weak Law of Large Numbers it follows that B^N/N converges to \bar{b} in probability, i.e., that

$$\text{plim}_{N \rightarrow \infty} \frac{B^N}{N} = \bar{b} \quad (2.20)$$

Hence for large populations the daily rate of job creation per-worker can be treated as nonstochastic.⁸

2.3.2. Unemployment Rate

A similar argument can be made for the steady-state unemployment level in the economy. In particular, if the unemployment status of each worker i is represented

⁸Note that our assumptions imply strong convergence in (2.20), i.e. that B^N/N converges almost surely to \bar{b} . But since weak convergence ensures the validity of nonstochastic approximations for any (sufficiently large) fixed population size, N , this suffices for our purposes.

by a random (indicator) variable, U_i (with $U_i = 1$ when i is unemployed and $U_i = 0$ otherwise), then the daily level of unemployment can be represented as in (2.19) by

$$U^N = \sum_{i=1}^N U_i \quad (2.21)$$

Hence if large populations of workers are treated as homogeneous collections of independent behaving units, then these employment status variables can be treated as independently and identically distributed with common mean, u [and finite variance $u(1-u)$]. It then follows, as in (2.20), that

$$\text{plim}_{N \rightarrow \infty} \frac{U^N}{N} = u \quad (2.22)$$

and thus that for large populations the steady-state unemployment rate, u , can also be regarded as nonstochastic.⁹

2.3.3. Job Vacancy rate

These nonstochastic approximations in turn imply that the steady-state job vacancy rate can be treated as nonstochastic. In particular, if the random variable J^N represents total jobs in steady state, so that by definition,

$$J^N = V^N + N(1-u) \quad (2.23)$$

it can then be shown that our assumptions imply

$$\text{plim}_{N \rightarrow \infty} \frac{J^N}{N} = \lim_{N \rightarrow \infty} \frac{E J^N}{N} = \frac{1}{2} \quad (2.24)$$

Thus, by dividing through (2.23) and taking probability limits, we may conclude that

$$\begin{aligned} \text{plim}_{N \rightarrow \infty} \frac{J^N}{N} &= \text{plim}_{N \rightarrow \infty} \left(\frac{V^N}{N} + (1-u) \right) \\ \frac{1}{2} &= \text{plim}_{N \rightarrow \infty} \frac{V^N}{N} + (1-u) \\ \text{plim}_{N \rightarrow \infty} \frac{V^N}{N} &= \frac{1}{2} - (1-u) \end{aligned} \quad (2.25)$$

⁹The assumption of statistically independent worker behavior can be relaxed to some degree. For example, if workers communicate only with a boundedly finite set of other workers, then a simple application of Chebyshev's Inequality shows that (2.22) still holds. More generally, the same arguments show that it is enough to require that the average (absolute) correlation, $(1/N) \sum_{j \neq i} \rho_{ij}$, of each worker i 's unemployment status with all other workers eventually vanish [see for example the proof of Theorem 4.4.2 of Renyi (1970)]. Similar (but more stringent) conditions for almost sure convergence can be found in Stout (1974, section 3.7).

Hence the steady-state job vacancy rate, v , can also be treated as nonstochastic for large N , and is seen to be given by

$$v = u + a \quad (2.26)$$

where

$$a = \frac{1}{2} \left(1 - \frac{1}{N} \right) \quad (2.27)$$

2.3.4. Vacancy Filling Rate

Finally, these results taken together also imply that the steady-state rate at which vacant jobs are filled can also be treated as nonstochastic. To see this, observe first that if for each $x, y; N$ we let

$$p_N(x; y) = \frac{1}{N} \left(\frac{y}{x} \right)^{1/N} \quad (2.28)$$

then the job-filling probability in (2.10) can be rewritten as follows:

$$\begin{aligned} P(F^N = j | U^N = n; m) &= \sum_{i=0}^j p_N(x; y) \binom{m}{i} \left(\frac{1}{N} \right)^i \left(\frac{N-1}{N} \right)^{m-i} \\ &= \sum_{i=0}^j \binom{m}{i} \left(\frac{1}{N} \right)^i \left(\frac{N-1}{N} \right)^{m-i} \\ &= p_N \left(\frac{n}{N}; \frac{m}{N} \right) \binom{m}{j} \left(\frac{1}{N} \right)^j \left(\frac{N-1}{N} \right)^{m-j} \end{aligned} \quad (2.29)$$

In terms of this notation, observe next from (2.4) that the total number, F^N , of job vacancies filled must then be conditionally binomially distributed given $U^N = n$ and $V^N = m$, with mean

$$E(F^N | U^N = n; m) = p_N \left(\frac{n}{N}; \frac{m}{N} \right) m \quad (2.30)$$

Hence, the conditional expectation of the vacancy-filling rate, F^N/N , is

$$E \left(\frac{F^N}{N} | U^N = n; m \right) = p_N \left(\frac{n}{N}; \frac{m}{N} \right) \frac{m}{N} \quad (2.31)$$

and it follows that the unconditional mean of F^N/N is given by

$$E \left(\frac{F^N}{N} \right) = E_{U^N, V^N} \left[p_N \left(\frac{U^N}{N}; \frac{V^N}{N} \right) \frac{V^N}{N} \right] \quad (2.32)$$

But if we now let

$$p(x; y) = \int_0^1 e^{i(sx-y)} \quad (2.33)$$

and observe that

$$\begin{aligned} \lim_{N \rightarrow \infty} p_N(x; y) &= \lim_{N \rightarrow \infty} \int_0^1 e^{i(sx-y)} \frac{1}{N} ds; \\ &= \int_0^1 e^{i(sx-y)} ds \\ &= p(x; y) \end{aligned} \quad (2.34)$$

it can then be shown that this limiting relation, together with the probability limits in (2.22) and (2.26) imply that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{F^N}{N} &= \lim_{N \rightarrow \infty} E \frac{F^N}{N} \\ &= \lim_{N \rightarrow \infty} E_{U^N, V^N} p_N \left(\frac{U^N}{N}; \frac{V^N}{N} \right) \\ &= p(u; v) \int_0^1 e^{i(su-v)} ds \end{aligned} \quad (2.35)$$

If the steady-state vacancy-filling rate, f , for the system is defined by

$$f = \lim_{N \rightarrow \infty} \frac{F^N}{N}; \quad (2.36)$$

then it follows from (2.35) that f is related to v and u by

$$f = \hat{A}(u; v) = v \int_0^1 e^{i(su-v)} ds; \quad (2.37)$$

The function, \hat{A} ; is seen to be precisely the asymptotic (normalized) form of the functions, \hat{C}_N , defined in (2.12) above, and hence is again designated as the matching function for the system. Notice also from (2.29) and (2.34) that the quantity

$$p_f(u; v) = \int_0^1 e^{i(su-v)} ds \quad (2.38)$$

can be interpreted as the asymptotic job-filling probability for the system (i.e. the asymptotic probability that any given vacant job will be filled on a given day).

Finally we note that while the above analysis was developed in terms of job-filling probabilities, all results could equivalently have been formulated in terms

of hiring probabilities. This follows from the obvious accounting identity between the number of jobs filled and the number of workers hired on any given day. Therefore, denoting the desired asymptotic hiring probability for the system by

$$p_h(u; v) = \text{plim}_{N \rightarrow \infty} q_N \frac{U^N}{N}; \frac{V^N}{N} \quad (2.39)$$

it follows from (2.35) that

$$p_h(u; v) = \frac{v}{s} \prod_{i=1}^h e^{i(su-v)} \quad (2.40)$$

2.4. Properties of the Matching Function

Before determining the steady state values of u and v , it is of interest to consider the matching function (2.37) in more detail. First, we show that \bar{A} is an instance of the general class of matching functions proposed by Pissarides (1990), i.e. that

Proposition 2.1. The function, \bar{A} ; in (2:37) is concave, linearly homogeneous, and monotone increasing in both arguments.

Proof. See section A.1 of the Appendix.

We now compare the matching function (2.37) with its finite-population counterpart in (2.12) above. To do so, it is convenient to consider the normalized form, \bar{A}_N , of \bar{A}_N as defined in (2.31) [together with (2.28)] above for all $u = n/N$ and $v = m/N$ by

$$\begin{aligned} \bar{A}_N(u; v) &= E \frac{\bar{A}_N(n; m)}{N} = \frac{m}{N} \prod_{i=1}^m \left(\frac{n}{N} + \frac{m}{N} \right)^{i-1} \\ &= v \prod_{i=1}^m \left(\frac{u}{v} + 1 \right)^{i-1} \frac{1}{N} \end{aligned} \quad (2.41)$$

If u and v are treated as continuous variables, then partial differentiation of (2.41) again shows that $\bar{A}_N(u; v)$ is increasing and concave in the relevant range of each variable (i.e., for all values of $u \in [0, 1]$ and $v \in [0, 1]$). However, it should be clear from the form of (2.41) that \bar{A}_N fails to be linearly homogeneous. Indeed, it is clear from the argument above that linear homogeneity is essentially a consequence of limits. To see this, first observe that \bar{A}_N is of the form

$$\bar{A}_N(u; v) = v \prod_{i=1}^m P(Nu; Nv) \quad (2.42)$$

where $P(x; y) = \prod_{i=1}^x \left(\frac{x}{y} + 1 \right)^{i-1}$. More generally, suppose that \bar{A}_N is of the form (2.42) for any choice of P for which a limiting value

$$p(u; v) = \lim_{N \rightarrow \infty} \prod_{i=1}^m P(Nu; Nv) \quad (2.43)$$

exists, so that the desired asymptotic matching function can be written obtained as

$$\hat{A}(u; v) = v \zeta \gg(u; v) = v \zeta \lim_{N! \rightarrow 1} P(Nu; Nv) \quad (2.44)$$

Then for any $\zeta > 0$ it follows from (2.43) that

$$\begin{aligned} \gg(\zeta u; \zeta v) &= \lim_{N! \rightarrow 1} P[N(\zeta u); N(\zeta v)] \\ &= \lim_{N! \rightarrow 1} P[(N\zeta)u; (N\zeta)v] \\ &= \lim_{(N\zeta)! \rightarrow 1} P[(N\zeta)u; (N\zeta)v] \\ &= \gg(u; v) \end{aligned} \quad (2.45)$$

Hence the limit probability function, \gg , must be homogeneous of degree zero, and it follows at once from (2.44) that \hat{A} must be homogenous of degree one, i.e. that

$$\hat{A}(\zeta u; \zeta v) = \zeta v \zeta \gg(\zeta u; \zeta v) = \zeta v \zeta \gg(u; v) = \zeta \hat{A}(u; v) \quad (2.46)$$

Note finally that this limit property also shows that any finite-population matching function of the form (2.42) must always be approximately linearly homogeneous as N becomes large. In the present case, it is well known that the limit in (2.34) is reached rather quickly. In fact, numerical examples show that for populations as small $N = 5$ the function \hat{A}_N in (2.41) is approximately linearly homogeneous for all $\zeta > 1$:

2.5. System Steady States

Given these asymptotic approximations, the appropriate equations describing steady states of the system can now be summarized as follows. Observe first from the limiting relations

$$(u; v; f) = \text{plim}_{N! \rightarrow 1} \left[E \frac{\tilde{U}^N}{N}; E \frac{\tilde{V}^N}{N}; E \frac{\tilde{F}^N}{N} \right] \quad (2.47)$$

that by dividing both (2.17) and (2.18) by N and taking limits, we obtain the steady-state relations

$$(1 - \frac{1}{2}) f + \frac{1}{2} v = \bar{c} \quad (2.48)$$

$$(1 - \frac{1}{2}) f = \frac{1}{2} (1 - u) \quad (2.49)$$

which together with (2.37) can be transformed into the pair of steady-state equations¹⁰

$$v + (1 - u) = \bar{c} = \frac{1}{2} \quad (2.50)$$

¹⁰Notice that (2.50) is identical with the steady-state relation already obtained in (2.26).

$$\frac{1}{2} (1 - \beta) u = (1 - \beta) v^h \frac{1}{2} e^{i(su=v)} \quad (2.51)$$

If for any given search intensity, $s \in [0; 1]$, we now designate each solution, $[u(s); v(s)]$, to [(2.50), (2.51)] for [(2.26); (2.51)] as a steady state for s , then the main result of this section is to establish the existence and uniqueness of steady states. To do so, note first that when $\beta < \frac{1}{2}$ [so that by (2.24) there is less than one job per person in the steady state], it is possible to obtain negative values for vacancy rates, v , in (2.50). But since steady states are only meaningful if $v \geq 0$, we now specify this nonnegativity condition in terms of unemployment rates, u , as follows. Recall from (2.26) that $v = u + a \geq 0 \Leftrightarrow u \geq -a$. Hence letting

$$u_a = \max\{0; -a\}; \quad (2.52)$$

it follows that the only meaningful steady states are those with $u \in [u_a; 1]$. With these observations, our main result is to show that (the proof is given in section A.5 of the Appendix):

Theorem 2.2 (Fixed-Search Equilibrium). For each search intensity, $s \in (0; 1]$, there exists a unique steady state, $[u(s); v(s)]$, with $u(s) \in [u_a; 1]$. In addition, both $u(s)$ and $v(s)$ are positive decreasing differentiable functions of s with $u(0) = 1$ and $v(0) = \beta - \frac{1}{2}$.

Proof. See section A.2 of the Appendix.

This steady state can be compared to that of the MP (Mortensen-Pissarides) model, by noting that the Beveridge curve in MP is very similar to our steady-state condition (2.51). In contrast to the free entry condition in MP (mentioned in the Introduction), our steady-state condition (2.50) results from the underlying birth-death process on vacancies. This steady-state can also be compared to that of the BD (Blanchard-Diamond) model by observing that equations (2.50) and (2.51) are essentially a variation of the steady-state conditions derivable from their recursive equation system [(A.1) through (A.5)].¹¹ The only real differences (aside from stochasticity) are that (i) their discrete time period is a week rather than a day, (ii) their job seekers are indexed by unemployment duration, and (iii) their stock of possible jobs is finite rather than denumerable, as in our case.¹²

¹¹ In particular, if $\frac{1}{2}$ is replaced by the transition probability $\frac{1}{4}$ in BD, then our condition (2.51) is a consequence of the steady-state conditions derivable from conditions [(A1), (A3), (A5)] in BD. Similarly, if our steady-state expected jobs per worker, $\beta - \frac{1}{2}$, is replaced by their steady-state expected jobs per worker, $\frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} \frac{K}{N}$, then our condition (2.50) is a consequence of the steady-state conditions derivable from conditions [(A1), (A2), (A3), (A5)] in BD.

¹² Recall that our basic states are of the form $(V_t^N = m; U_t^N = n)$ with implicit domains $m \in \mathbb{Z}_+$ and $n \in \{0; 1; \dots; N\}$. Note also that the finite-stock model in BD requires that each job have three possible states: 'filled', 'vacant', and 'idle'. However this distinction is of little consequence in terms of the actual behavior of the two models.

It can then easily be shown that the standard comparative statics results on u and v (see e.g. Pissarides, 1990) are obtained as well as the following ones:

$$\frac{\partial u}{\partial \theta} < 0 \quad \frac{\partial u}{\partial \tau} < 0 \quad \frac{\partial u}{\partial \lambda} > 0 \quad (2.53)$$

$$\frac{\partial v}{\partial \theta} < 0 \quad \frac{\partial v}{\partial \tau} > 0 \quad \frac{\partial v}{\partial \lambda} < 0 \quad (2.54)$$

These relations can be used to further clarify the basic role of each parameter in the model. From the unemployed worker's viewpoint (2.53) it is clear that he/she is more likely to get a job by being qualified for a wider range of jobs (θ). Similarly, hiring is more likely if new job opportunities are created at a faster rate (τ) or existing job opportunities are closed at a slower rate (λ). From the firm's viewpoint (2.54) it is equally clear that increased broader worker qualifications (θ) will increase the likelihood of a job being filled, and hence decrease vacancies. Also, vacancies are more likely to be filled if new vacancies are created at a slower rate (τ) or if existing vacancies die at a faster rate (λ). Note in particular, that the qualification probability, θ , can now be viewed as a skill matching index in the sense that larger values increase both the likelihood of unemployed workers being hired and vacant jobs being filled. Thus, our model explicitly introduces the parameter θ for 'skill mismatch' whereas it is usually introduced in an ad hoc fashion in the empirical literature.

3. Endogenous Search Intensities

These observations lead naturally to the question of how unemployment levels may be affected if unemployed workers are allowed to choose their own levels of search intensity. Basically, this choice involves a trade-off between the leisure time lost and the expected income gained (from quicker job acquisition) by more frequent job search. To model this trade-off explicitly, we begin by considering the relative value of employment and unemployment in terms of the utility levels and discounted utility streams associated with each state. Here the respective classes of unemployed workers and employed workers are denoted by '0' and '1'.

3.1. Effective-Income Utility

If individual daily income is denoted by y , and if the fraction of time spent each day in leisure activities is denoted by l , then the individual's utility for $(l; y)$ is postulated to be of the form

$$U(l; y) = l^\alpha y \quad (3.1)$$

with parameter $\alpha \in (0; 1)$. This utility can be regarded as the individual's effective daily income, discounted by the fraction of leisure time available for consumption.¹³ As will become clear below, the assumption of decreasing returns to leisure (i.e., $0 < \alpha < 1$) is critical for our purposes, in that it allows meaningful optimal search intensities, s , other than the extremes 0 and 1. In particular, if the leisure time for unemployed workers is taken to include all time spent not searching for work, then the expected fraction of leisure time for an unemployed worker with search intensity, s , is simply $1 - s$. Hence if it is assumed that income for such individuals is given by a daily unemployment benefit, b , then the relevant effective daily income for each unemployed worker is given by

$$U_0(s) = (1 - s)^\alpha b \quad (3.2)$$

Here search intensity, s , constitutes the only relevant decision variable for unemployed workers.

Similarly, if it is assumed that the income of employed workers is given by the daily wage, w , and that their fraction of leisure time, l_1 , is constant, then the effective daily income for all employed workers is given by

$$U_1 = l_1^\alpha w \quad (3.3)$$

Since the wage level is here taken to be fixed, there are no relevant decision variables for employed workers. Hence the value U_1 can be regarded as an exogenous parameter in the present model.

3.2. Lifetime Effective-Income Streams

Recall that our basic objective is to model the decision problem for an unemployed worker who is currently considering his/her choice of search intensity, s , (which for simplicity can be regarded as the choice of a roulette wheel to use each morning in deciding whether to search that day). To weigh alternative choices, the worker must evaluate the expected future effective-income streams resulting from each choice of s . [In the following development, we suppress dependence of all variables on s except when needed, so that for example we write U_0 for $U_0(s)$.] At each

¹³Note also that this utility can be viewed as the indirect utility obtained from a standard log-linear function, $U(l; z) = l^\alpha z^{1-\alpha}$, in leisure time, l , and composite good, z , subject to time and budget constraints. In particular, if the price of z is taken to be the numeraire, and l is treated as the fraction of time spent in leisure (so that total time is one), then the budget and time constraints can be written respectively as $z = y$ and $l + s = 1$ so that $0 < l < 1$. Hence the corresponding indirect utility (with respect to the budget constraint) is obtained by replacing z with y . The choice of $\alpha = 1$ (together with the dimensionless nature of l) yields an indirect utility in (3.1) which is in monetary units, and hence is interpretable as 'effective' income.

point of time in the future the worker will be in one of two states: unemployed (0) or employed (1). If the relevant discount rate for all workers is the same, and is denoted by $\frac{1}{1+i}$ (0; 1) [representing the value today of a dollar received tomorrow], then the expected values, $E(I_0)$ and $E(I_1)$, of the discounted effective income streams, I_0 and I_1 , starting from each possible state can be determined as follows. Observe that if the duration times (number of consecutive days) in each state are denoted respectively by T_0 and T_1 , then by employing the identity, $\sum_{k=0}^{t-1} \frac{1}{1+i} \left(\frac{1}{1+i}\right)^k = \frac{1}{1+i} \left(\frac{1-i^{-t}}{1-i^{-1}}\right) = \frac{1-i^{-t}}{i}$, it follows that the conditional expectation of I_0 given a duration of t days in unemployment must be of the form:

$$\begin{aligned} E(I_0 | T_0 = t) &= \sum_{k=0}^{t-1} \frac{1}{1+i} \left(\frac{1}{1+i}\right)^k U_0 + \frac{1}{1+i} \left(\frac{1}{1+i}\right)^t E(I_1) \\ &= \frac{1-i^{-t}}{i} U_0 + \frac{1}{1+i} \left(\frac{1}{1+i}\right)^t E(I_1) ; t = 1; 2; \dots \end{aligned} \quad (3.4)$$

For example, workers hired on the first day do not start work until the next day (by assumption). This implies that on the first day, workers continue to receive unemployment benefits, b , and realize utility level, U_0 . From the next day onward, workers will enjoy the expected utility stream, $E(I_1)$, starting in the employed state, so that $E(I_0 | T_0 = 1) = U_0 + \frac{1}{1+i} E(I_1)$.

Similarly, when a worker is employed, the conditional expectation of I_1 given an employment duration of t days is easily seen to be of essentially the same form:

$$E(I_1 | T_1 = t) = \frac{1-i^{-t}}{i} U_1 + \frac{1}{1+i} \left(\frac{1}{1+i}\right)^t E(I_0) ; t = 0; 1; 2; \dots \quad (3.5)$$

Notice however that in this case, employment duration starts from zero rather than one. This is a consequence of our assumption that a worker may be hired and let go on the same day. In this particular case, the expected utility stream is not different from that of an unemployed worker, so that $E(I_1 | T_1 = 0) = (0)U_1 + \frac{1}{1+i} E(I_0) = E(I_0)$. On the other hand, employed workers who are not laid off immediately will enjoy at least one day of effective employment income, U_1 .

If we now identify the lifetime values, V_0 and V_1 , of these states with the unconditional expectations, $E(I_0)$ and $E(I_1)$,¹⁴ then we may employ (3.4) and (3.5) to solve for these values in terms of the effective incomes, U_0 and U_1 , as follows. First observe that by definition,

$$V_0 = E(I_0) = E_{T_0} [E(I_0 | T_0)] = \frac{1-i^{-e_0}}{i} U_0 + \frac{1}{1+i} \left(\frac{1}{1+i}\right)^{e_0} V_1 \quad (3.6)$$

¹⁴As with all models of this type, workers are assumed to ignore their own mortality, and to behave as though their future were infinite.

where $e_0 = E \left[\sum_{t=0}^{\infty} \frac{1}{4^t} \right]$, and similarly that,

$$V_1 = E(I_1) = E_{T_1} [E(I_1 | T_1)] = \frac{1 - e_1}{1 - \frac{3}{4}} U_1 + e_1 V_0 \quad (3.7)$$

where $e_1 = E \left[\sum_{t=1}^{\infty} \frac{1}{4^t} \right]$. These equations may in turn be solved simultaneously to yield the following expressions for V_0 and V_1 in terms of U_0 and U_1 :

$$V_0 = \frac{1 - e_0}{1 - \frac{3}{4} - e_0 e_1} \frac{1}{4} U_0 + \frac{e_0 (1 - e_1)}{1 - \frac{3}{4} - e_0 e_1} \frac{1}{4} U_1 \quad (3.8)$$

$$V_1 = \frac{1 - e_1}{1 - \frac{3}{4} - e_0 e_1} \frac{1}{4} U_1 + \frac{e_1 (1 - e_0)}{1 - \frac{3}{4} - e_0 e_1} \frac{1}{4} U_0 \quad (3.9)$$

What remains to be determined are the expected discount factors e_0 and e_1 . To do so, we begin by establishing the exact distributions of T_0 and T_1 . Turning first to T_0 , let the hiring probability in (2.40) be denoted by p_h , and observe that the probability of leaving unemployment on day $t = 1$ is by definition the joint probability, $s p_h$, of going to the labor market on that day and being hired. Moreover, since an unemployment duration of t days means precisely that this joint event first occurs on day t , it follows from our independence assumptions that T_0 must be geometrically distributed according to

$$P(T_0 = t) = (1 - s p_h)^{t-1} s p_h; \quad t = 1; 2; \dots \quad (3.10)$$

Next, to determine the distribution of T_1 , recall that the probability of job termination on any day is given by $\frac{1}{2}$. Hence it follows from our 'birth-death' assumptions on jobs that T_1 must also be geometrically distributed according to¹⁵

$$P(T_1 = t) = (1 - \frac{1}{2})^t \frac{1}{2}; \quad t = 0; 1; 2; \dots \quad (3.11)$$

Given these two distributions, we may now compute the desired expectations as follows. First, by employing the identity, $\sum_{t=1}^{\infty} a^t = a / (1 - a)$ for all $a \in [0; 1)$, it follows from (3.10) that

$$e_0 = E \left[\sum_{t=0}^{\infty} \frac{1}{4^t} \right] = \frac{\frac{3}{4} p_h s}{1 - \frac{3}{4} + \frac{3}{4} p_h s} \quad (3.12)$$

Similarly, by employing the identity, $\sum_{t=0}^{\infty} a^t = 1 / (1 - a)$ for all $a \in [0; 1)$, it follows from (3.11) that

$$e_1 = E \left[\sum_{t=1}^{\infty} \frac{1}{4^t} \right] = \frac{\frac{1}{2}}{1 - \frac{3}{4} + \frac{3}{4} \frac{1}{2}} \quad (3.13)$$

¹⁵In the discrete time version of standard birth-death processes, exponential lifetimes are replaced by their discrete geometric counterparts.

3.3. Calculation of the Optimal Search Intensity

Returning to our basic decision problem, suppose that an unemployed worker is currently reconsidering his search intensity level, s . With an eye toward our ultimate goal of determining an equilibrium search intensity for the system, suppose also that the system is in a steady state where all workers are currently using the same search intensity level, say \bar{s} . Associated with this search intensity level (as in section 1 above) is a steady-state hiring probability (2.40) which we again denote by $p_h = p_h(\bar{s})$, and a steady-state lifetime value of employment, which we denote by $V_1 = V_1(\bar{s})$. Here it is assumed that perturbations in the search intensity, s , of a single unemployed worker cannot influence these steady state values, and hence that p_h and V_1 can be treated as constants in the worker's decision problem. To formulate this problem, observe next that the expected discount value, e_0 , in (3.12) now takes the form

$$e_0(s) = \frac{\frac{3}{4} p_h s}{1 - \frac{3}{4} + \frac{3}{4} p_h s} \quad (3.14)$$

where the hiring probability, p_h , is again beyond the individual's control, but where his choice of s affects his unemployment duration, and thus his value of e_0 . Given these observations, we may now use (3.2), (3.6) and (3.12) to express his lifetime value, V_0 , solely in terms of s as follows:

$$V_0(s) = \frac{1 - e_0(s)}{1 - \frac{3}{4}} U_0(s) + e_0(s) V_1 = \frac{(1 - s)^b + \frac{3}{4} p_h s V_1}{1 - \frac{3}{4} + \frac{3}{4} p_h s} \quad (3.15)$$

Thus the relevant decision problem is to choose a value of $s \in [0, 1]$ which maximizes (3.15). To solve this problem, we begin by differentiating (3.15). We obtain:

$$V_0'(s) = \frac{1}{1 - \frac{3}{4} + \frac{3}{4} p_h s} \left[\frac{3}{4} p_h V_1 - (1 - s)^{b-1} b \frac{3}{4} p_h V_0(s) \right] \quad (3.16)$$

Observe that the first-order condition, $V_0'(s) = 0$, holds if the bracketed term is zero. This can be rewritten as

$$(1 - s)^{b-1} b = \frac{3}{4} p_h [V_1 - V_0(s)] \quad (3.17)$$

The interpretation of (3.17) is straightforward. The left hand side is the short-run utility loss from a marginal increase in search intensity, and the right hand side is the corresponding long-run utility gain from future employment. Thus, the level of search intensity is optimal when the marginal gain of searching (reduced unemployment duration) is equal to its marginal cost (reduced leisure time). By totally differentiating (3.17), we obtain the following comparative-statics results (where p_h and V_1 are here taken to be parameters):

$$\frac{\partial s}{\partial b} < 0 \quad \frac{\partial s}{\partial p_h} > 0 \quad \frac{\partial s}{\partial V_1} > 0 \quad \frac{\partial s}{\partial \frac{3}{4}} > 0 \quad (3.18)$$

These lend support to the observations above. When there is an increase in unemployment benefits (b), working becomes less attractive and unemployed workers are less motivated to search. On the other hand, when there is an increase in either the chance of being hired (p_h), the value of being employed (V_1), or the importance of the future (β), unemployed workers are more motivated to search.

To establish the existence of solutions to (3.17) observe next from (3.16) that $\lim_{s \rightarrow 1} V_0^0(s) = \beta - 1$. Hence a sufficient condition for at least one solution to (3.17) in $(0, 1)$ is that $V_0^0(0) > 0$. Moreover, by differentiating (3.16) once more, we see that

$$V_0^{00}(s) = \frac{\beta(1-\beta)(1-\beta s)^{\beta-2} \beta - 2\beta p_h V_0^0(s)}{1-\beta + \beta p_h s} \quad (3.19)$$

and hence that

$$V_0^0(s) = 0 \Rightarrow V_0^{00}(s) = \frac{\beta(1-\beta)(1-\beta s)^{\beta-2} \beta - 2\beta p_h V_0^0(s)}{1-\beta + \beta p_h s} < 0 \quad (3.20)$$

Thus (3.15) can have at most one interior maximum, and we conclude that:

Proposition 3.1. For any given population search intensity, \bar{s} , satisfying the condition that $V_0^0(0) > 0$, there exists a unique individual-optimum search intensity, $s(\bar{s})$. Moreover, this optimal search intensity is always in the open interval $(0, 1)$, and is given by the unique solution to (3.17).

3.4. Equilibrium with Endogenous Search Intensities

This solution to the optimal-search-intensity problem for unemployed workers leads directly to an equilibrium condition for the system as a whole. In particular, when search intensities are allowed to be endogenous, it is clear that a population search intensity level, \bar{s} , is a (Nash) equilibrium for the system if \bar{s} is the optimal individual response to itself, i.e., if $\bar{s} = s(\bar{s})$. Hence, if we now drop the 'bar' notation, and designate each common population choice of s as a population search intensity, then it follows from (3.17) that a population search intensity, s , is an equilibrium for the system if s satisfies the following condition:

$$\beta(1-\beta s)^{\beta-1} b = \beta p_h(s) [V_1(s) - V_0(s)] \quad (3.21)$$

where the hiring probability, $p_h(s)$, and the lifetime value of employment, $V_1(s)$, are now written as functions of the population search intensity, s . Note however that this equilibrium condition assumes that s is positive, which in view of Proposition 3.1 is equivalent to assuming that $V_0^0(0) > 0$. Moreover, the functions $p_h(s)$

and $V_1(s)$ are not expressible in closed form, but rather are implicit functions of s which depend on many other equilibrium quantities [including the steady-state unemployment rate, $u(s)$; and the equilibrium expected discount rate, $e_0(s)$]. Hence our next objective is to give an exact formulation of the desired equilibrium.

First we give an appropriate parametric specification of the positivity condition, $V_0^0(0) > 0$. It is easy to show that this condition is equivalent to:

$$U_1 > 1 + \frac{\mu(1 - \beta/2 + \beta/4)}{\beta/4 - (1 - \beta/2)} b; \quad (3.22)$$

which we now designate as the positivity condition. To interpret this condition, recall that since the unemployment benefit, b , is precisely the effective income of an unemployed worker, this positivity condition simply requires that the effective wage income of employment [in (3.3)] be sufficiently greater than the unemployment benefit to encourage some degree of job search. Note that the simpler condition that U_1 be greater than b is by itself not sufficient, precisely because the lifetime value of employment necessarily involves some time spent in unemployment. Note also that this basic positivity condition involves all parameters of the present model except for the job-creation rate, β . The key effect of this parameter is on the hiring probability, which (as we have seen) reduces to $\beta/4$ as s approaches zero. Hence the only condition on β needed to ensure positive equilibrium search intensity is that $\beta > 0$, i.e., that some vacant jobs be available to potential searchers.

Given this basic positivity condition, we now gather together all variables and conditions which define the desired equilibrium. For any positive parameter vector $(\mu; \beta; \beta/2; \beta/4; b; U_1)$ satisfying condition (3.22) with $\mu; \beta/2; \beta/4 \in (0; 1)$, a positive vector $E = (s; u; v; p_h; e_0; V_0; V_1)$ is designated as an endogenous-search equilibrium if E satisfies the following seven conditions with $s \in (0; 1)$ [where $a = (\beta/2 - \beta/4) - 1$; $e_1 = \beta/2 - (1 - \beta/4 + \beta/4)$ and $U_0(s) = (1 - s)\mu b$]:

$$v = u + a \quad (3.23)$$

$$p_h = \frac{v}{s u} (1 - \beta/2) e^{i - \beta/4} \quad (3.24)$$

$$\beta/2 (1 - u) = (1 - \beta/2) u s p_h \quad (3.25)$$

$$e_0 = \frac{\beta/4 p_h s}{1 - \beta/4 + \beta/4 p_h s} \quad (3.26)$$

$$V_0 = \frac{\mu(1 - \beta/4) e_0}{1 - \beta/4} U_0(s) + e_0 V_1 \quad (3.27)$$

$$V_1 = \frac{\mu(1 - \beta/4) e_1}{1 - \beta/4} U_1 + e_1 V_0 \quad (3.28)$$

$$\frac{3}{4} p_h (V_1 - V_0) = \theta (1 - s)^{\theta-1} b \quad (3.29)$$

The first three conditions follow directly from the steady-state equations (2.50) and (2.51) together with the definition of the hiring probability in (2.40). Conditions (3.26), (3.27), and (3.28) are precisely the definitions of e_0 , V_0 , and V_1 in (3.14), (3.6), and (3.7), respectively. Finally, (3.29) is a restatement of the basic first-order condition in (3.21). Given this formal definition, our main result is to show that (the proof is given in section A.8 of the Appendix):

Theorem 3.2 (Endogenous-Search Equilibrium). For each positive parameter vector $(\theta; \tau; \frac{1}{2}; \rho; \frac{3}{4}; b; U_1)$ satisfying condition (3.22) with $\theta; \frac{1}{2}; \rho; \frac{3}{4} \in (0, 1)$, there exists a unique endogenous-search equilibrium.

Proof. See section A.3 of the Appendix.

4. Concluding Remarks

In this paper, we have presented an explicit micro model of job matching in which heterogeneous workers allocate themselves to jobs with different skill requirements. Within this framework it was shown that when the population size is large, the asymptotic form of aggregate matching function is of an exponential type, and is an instance of the general class of 'production-like' matching functions described by Pissarides (1990). This function is of course only one among many possibilities, including the Cobb-Douglas matching function used most frequently in empirical research. Hence its major advantage is that it is derivable directly from an explicit micro scenario which captures many important aspects of job matching behavior. However, it is equally clear that this scenario is in many ways too simplistic, and for example assumes both identical wages and revenues for all jobs and identical information levels and matching (qualification) probabilities for all job seekers. Hence this model is perhaps best regarded as a benchmark for constructing more realistic behavioral scenarios. In the second part of the paper, it is then shown how this model can be used to analyze the decision behavior of unemployed workers in choosing their optimal search intensities, i.e., how many days per week to search. We prove that this model is again consistent with a unique steady state.

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A. Appendix

A.1. Proof of Proposition 2.1

First observe that for all $s > 0$;

$$\hat{A}(s, u; s, v) = s v^{-1} e^{i(s, u=s, v)} = s v^{-1} e^{i(su=v)} = s \hat{A}(u; v)$$

and hence that \hat{A} is linearly homogeneous. Next observe that \hat{A} is twice continuously differentiable with:

$$\frac{\partial}{\partial u} \hat{A}(u; v) = s e^{i(su=v)} \quad (\text{A.1})$$

$$\frac{\partial^2}{\partial u^2} \hat{A}(u; v) = i \frac{(s)^2}{v} e^{i(su=v)} \quad (\text{A.2})$$

$$\frac{\partial}{\partial v} \hat{A}(u; v) = \frac{1}{v} \left(1 - \frac{su}{v} \right) e^{i(su=v)} \quad (\text{A.3})$$

$$\frac{\partial^2}{\partial v^2} \hat{A}(u; v) = i \frac{(su)^2}{v^3} e^{i(su=v)} \quad (\text{A.4})$$

so that monotonicity follows from the positivity of (A.1) and (A.3).¹⁶ Finally, to establish concavity, observe in addition that

$$\frac{\partial^2}{\partial v \partial u} \hat{A}(u; v) = -\frac{s}{v} \frac{su}{v} e^{i(su=v)} \quad (\text{A.5})$$

and hence from [(A.2), (A.4), (A.5)], that the Hessian of \hat{A} is given by

$$H = (s)^2 e^{i(su=v)} \begin{pmatrix} \frac{1}{v} & \frac{u}{v^2} \\ \frac{u}{v^2} & \frac{u^2}{v^3} \end{pmatrix} \quad (\text{A.6})$$

Thus for any column vector, $z = (x; y)^0$; it follows that

$$\begin{aligned} z^0 H z &= (s)^2 e^{i(su=v)} (x; y) \begin{pmatrix} \frac{1}{v} & \frac{u}{v^2} \\ \frac{u}{v^2} & \frac{u^2}{v^3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (s)^2 e^{i(su=v)} \left(\frac{1}{v} x^2 + 2 \frac{u}{v} xy + \frac{u^2}{v^3} y^2 \right) \\ &= i (s)^2 e^{i(su=v)} \frac{1}{v} x^2 + \frac{u}{v} y^2 \cdot 0 \end{aligned} \quad (\text{A.7})$$

and we see that H is negative semidefinite. Moreover, (A.7) is strictly negative except for vectors z colinear with $(u; v)$; i.e., with $z = (s, u; s, v)$ for some s : Hence \hat{A} is seen to be strictly concave except for the linearity-on-rays property implied by linear homogeneity. ■

¹⁶To see that (A.3) is positive, let $z(x) = 1 + (1+x)e^{-x}$; and observe that $z(0) = 0$ and $z'(x) = x e^{-x} > 0$ for all $x > 0$.

A.2. Proof of Theorem 2.2

By substituting (2.26) into (2.51) and letting

$$G(u; s) = \frac{1}{2} (1 - u) \left[(1 - \frac{1}{2})(u + a) - 1 \right] e^{i s \frac{u}{u+a}}; \quad (\text{A.8})$$

we see that for each given $s \in (0; 1]$ the steady-state values of u are precisely the roots of the equation

$$G(u; s) = 0 \quad (\text{A.9})$$

To establish existence of solutions, observe first that for $u = 1$ we have

$$G(1; s) = \frac{1}{2} (1 - \frac{1}{2})(1 + a) - 1 \left[e^{i s \frac{1}{1+a}} \right] \quad (\text{A.10})$$

But since $\frac{1}{2} > 0$ implies from (2.27) that $1 + a > 0$, we see that

$$G(1; s) < 0 \quad (\text{A.11})$$

Next we show that

$$G(u_a; s) > 0 \quad (\text{A.12})$$

for u_a in (2.52). To do so observe first that if $a > 0$ then $u_a = 0$ and

$$G(u_a; s) = \frac{1}{2} > 0; \quad (\text{A.13})$$

Next, if $a = 0$ then $G(u; s) = \frac{1}{2} (1 - u) \left[(1 - \frac{1}{2})u - 1 \right] e^{i s}$ and $u_0 = 0$, so again

$$G(u_a; s) = \frac{1}{2} > 0; \quad (\text{A.14})$$

Finally, if $a < 0$ then $\frac{1}{2} \geq 0$ implies that $u_a = |a| = |j a| < 1$, so $\lim_{u \rightarrow u_a} u = (u + a) = 1$ implies that $\lim_{u \rightarrow u_a} \left[(1 - \frac{1}{2})u - 1 \right] e^{i s \frac{u}{u+a}} = 1$; and hence that

$$\lim_{u \rightarrow u_a} G(u; s) = \frac{1}{2} (1 - |j a|) > 0 \quad (\text{A.15})$$

Thus (A.12) holds in all cases, and we may conclude from the continuity of G that for each $s \in (0; 1]$ there exists a steady-state value, $u(s)$, in the open interval $(u_a; 1)$. In particular, this implies that $u(s) > 0$. Next, to establish uniqueness of this steady state, it suffices to show the partial derivative of G with respect to u is everywhere negative in the interval $(u_a; 1)$, i.e. that

$$\frac{\partial}{\partial u} G(u; s) < 0; \quad u \in (u_a; 1) \quad (\text{A.16})$$

For this will imply that $G(u; s)$ can pass through zero no more than once in the interval $(u_a; 1)$: By using the identity

$$\frac{d}{du} \left[\frac{u}{u+a} \right] = \frac{a}{(u+a)^2} \quad (\text{A.17})$$

it may be verified that

$$\frac{\partial}{\partial u} G(u; s) = (1 - \frac{1}{2}) e^{i \circ s \frac{u}{u+a}} \frac{u + a - i \circ sa}{u + a} i - 1 \quad (A.18)$$

which together with $u + a > 0$ for all $u > u_a$ implies that (A.18) is well defined on $(u_a; 1)$; and in particular satisfies

$$\lim_{u \rightarrow 1} \frac{\partial}{\partial u} G(u; s) = (1 - \frac{1}{2}) e^{i \circ s} (1 - i) - 1 < 0 \quad (A.19)$$

Hence it suffices to show that the second partial derivative is positive for all $u > u_a$ [which will imply that (A.18) must be everywhere negative on $(u_a; 1)$]. By direct calculation it follows from (A.18) and (A.17) that

$$\frac{\partial^2}{\partial u^2} G(u; s) = \frac{(1 - \frac{1}{2})}{(u + a)^3} (\circ sa)^2 e^{i \circ s \frac{u}{u+a}} > 0 \quad (A.20)$$

for all $u > u_a$. Thus (A.16) holds, and $u(s)$ must be the unique solution of (A.8) in the interval $[u_a; 1]$. These unique solutions define a function of s which can be analyzed by implicit differentiation of $G[u(s); s]$ as follows. Since $G[u(s); s] \leq 0$; we see that

$$0 \leq \frac{\partial}{\partial s} G[u(s); s] = \frac{\partial G}{\partial u} u'(s) + \frac{\partial G}{\partial s} \\ \Rightarrow u'(s) \leq - \frac{\partial G / \partial s}{\partial G / \partial u} \quad (A.21)$$

for all $s \in [0; 1]$. But since

$$\frac{\partial}{\partial s} G(u; s) = -i (1 - \frac{1}{2}) \circ u e^{i \circ s \frac{u}{u+a}} < 0; \quad (A.22)$$

we may then conclude from (A.16) and (A.21) that $u'(s) < 0$, and hence that $u(s)$ is strictly decreasing in s . Finally since (2.26) implies that

$$v(s) = u(s) + a \quad (A.23)$$

for all s , and since $u(s) \in (u_a; 1)$ implies that $0 < u(s) + a = v(s)$, we see that $v(s)$ is also a decreasing positive differentiable function of s . ■

A.3. Proof of Theorem 3.2

To establish existence of a solution to equations (3.23) through (3.29), we first show that this system can be reduced to a single equation in search intensity, s , as follows. First observe from (3.26) and (3.27) that

$$\begin{aligned}
 V_0 &= \frac{(1 - e_0) U_0(s)}{1 - \frac{3}{4}} + e_0 V_1 \\
 &= \frac{1 - \frac{3}{4}}{1 - \frac{3}{4} + \frac{3}{4} p_h s} \frac{U_0(s)}{1 - \frac{3}{4}} + \frac{\frac{3}{4} p_h s}{1 - \frac{3}{4} + \frac{3}{4} p_h s} V_1 \\
) \quad (1 - \frac{3}{4} + \frac{3}{4} p_h s) V_0 &= U_0(s) + \frac{3}{4} p_h s V_1 \\
) \quad \frac{3}{4} p_h s (V_1 - V_0) &= (1 - \frac{3}{4}) V_0 - U_0(s) \tag{A.24}
 \end{aligned}$$

Also by (3.27) and (3.28),

$$\begin{aligned}
 V_0 &= (1 - e_0) \frac{U_0(s)}{1 - \frac{3}{4}} + e_0 \frac{1 - e_1}{1 - \frac{3}{4}} U_1 + e_1 V_0 \\
) \quad (1 - e_0 e_1) V_0 &= (1 - e_0) \frac{U_0(s)}{1 - \frac{3}{4}} + e_0 (1 - e_1) \frac{U_1}{1 - \frac{3}{4}} \\
) \quad (1 - \frac{3}{4}) V_0 &= \frac{1 - e_0}{1 - e_0 e_1} U_0(s) + \frac{e_0 (1 - e_1)}{1 - e_0 e_1} U_1; \tag{A.25}
 \end{aligned}$$

which together with (A.24) implies that

$$\begin{aligned}
 \frac{3}{4} p_h s (V_1 - V_0) &= \frac{1 - e_0}{1 - e_0 e_1} U_0(s) + \frac{e_0 (1 - e_1)}{1 - e_0 e_1} U_1 - U_0(s) \\
 &= \frac{e_0 (1 - e_1)}{1 - e_0 e_1} [U_1 - U_0(s)]; \tag{A.26}
 \end{aligned}$$

Hence by (3.29) and (A.26),

$$\begin{aligned}
 \frac{e_0 (1 - e_1)}{1 - e_0 e_1} [U_1 - U_0(s)] &= \frac{3}{4} p_h s (V_1 - V_0) = s (1 - s)^{\otimes} i^{\#} b^{\otimes} \\
 &= \frac{s}{1 - s} [(1 - s)^{\otimes} b^{\otimes}] = \frac{s}{1 - s} U_0(s)^{\otimes} \\
) \quad \frac{1}{s} \frac{1 - e_0 e_1}{1 - e_0 e_1} \frac{U_1 - U_0(s)}{U_0(s)} &= \frac{1 - e_0 e_1}{e_0 (1 - e_1)}; \tag{A.27}
 \end{aligned}$$

where the left hand side is now seen to be an explicit function of s . Next we show that the right hand side is an explicit function of the unemployment rate, u . To

do so, observe from (3.25) that

$$s p_h = \frac{\bar{A}^{\frac{1}{2}}}{1 - \frac{1}{2}} \frac{1 - u}{u} \quad (\text{A.28})$$

which together with (3.26) yields

$$e_0 = \frac{\frac{3}{4} \frac{1}{2} (1 - u)}{(1 - \frac{3}{4}) (1 - \frac{1}{2}) u + \frac{3}{4} \frac{1}{2} (1 - u)} : \quad (\text{A.29})$$

By substituting (A.29) into the right hand side of (A.27) and reducing, we obtain

$$\begin{aligned} \frac{1 - e_0 e_1}{e_0 (1 - e_1)} &= \frac{(1 - \frac{3}{4}) (1 - \frac{1}{2}) u + \frac{3}{4} \frac{1}{2} (1 - e_1) (1 - u)}{\frac{3}{4} \frac{1}{2} (1 - e_1) (1 - u)} \\ &= 1 + \frac{u}{1 - u} \end{aligned} \quad (\text{A.30})$$

where [recalling that $e_1 = \frac{1}{2} (1 - \frac{3}{4} + \frac{3}{4} \frac{1}{2})$] the constant μ is given by

$$\begin{aligned} \mu &= \frac{\frac{1}{4} \frac{1}{2} \frac{1}{2}}{\frac{3}{4} \frac{1}{2}} \frac{\bar{A}^{\frac{1}{2}}}{1 - \frac{1}{2}} \frac{1}{1 - e_1} \\ &= \frac{1 - \frac{3}{4} + \frac{3}{4} \frac{1}{2}}{\frac{3}{4} \frac{1}{2}} > 0 : \end{aligned} \quad (\text{A.31})$$

Thus, letting $W(s)$ be defined for all $s \in (0; 1]$ by

$$\begin{aligned} W(s) &= \frac{1}{s} \frac{\mu (1 - s)}{s} \frac{U_1 - U_0(s)}{U_0(s)} \\ &= \frac{1}{s} \frac{\mu (1 - s)}{s} \frac{U_1 - (1 - s)^b}{(1 - s)^b} \end{aligned} \quad (\text{A.32})$$

we see from (A.30) and (A.32) that (A.27) now takes the form, $W(s) = \mu [u = (1 - u)]$, so that u can be written in terms of s as

$$u = \frac{W(s)}{1 + W(s)} : \quad (\text{A.33})$$

Moreover, by substituting (3.23) and (3.24) into (3.25), we obtain the following additional equation in s and u :

$$\frac{1}{2} (1 - u) = (1 - \frac{1}{2}) \frac{u + a}{su} \frac{1}{1 - e^{-s \frac{u}{u+a}}} \quad (\text{A.34})$$

Thus, letting $G(u; s)$ again be defined as in (A.8), it follows by substituting (A.33) into (A.34) that this system can be reduced to single equation in s of the form

$$G \frac{W(s)}{1 + W(s)}; s = 0 \quad (\text{A.35})$$

To solve this equation, we next observe that (A.33) is only meaningful for values of s which yield positive values of u in (A.33) and v in (3.23). It can be shown (see Lemma A.1 in section A.4 of this Appendix) that for each possible value of a (i.e., $a > \frac{1}{2}$) there exists a unique search intensity, $s_a \in (0; 1)$, such that both these conditions hold $\forall s \in (0; s_a)$. With these observations, the key step in the proof (established in Lemma A.2 in section A.4 of this Appendix) is to show that for each $a > \frac{1}{2}$ there exists a unique solution to (A.35) in the open interval $(0; s_a)$.

Given this solution, s , it remains only to show that s generates a unique set of values $[u(s); v(s); p_h(s); e_0(s); V_0(s); V_1(s)]$ which together with s constitute an endogenous-search equilibrium. To do so, let $W(s)$ be defined by (A.32) and observe that if $u(s) \in (0; 1)$ is in turn defined by (A.33), and if we let $v(s) = u(s) + a$, then it follows from the definition of G that (3.23), (3.24) and (3.25) must hold for these choices, with $p_h(s)$ defined by

$$p_h(s) = \frac{v(s)^{\frac{1}{2}}}{s u(s)} (1 - e^{-s \frac{u(s)}{v(s)}})^{\frac{3}{4}} \quad (\text{A.36})$$

In particular, this is seen to imply that $[u(s); v(s)]$ constitutes a steady state given s , and hence (by Theorem 2.2) that $v(s) > 0$. Moreover, since $p_h(s)$ is of the form, $(1 - e^{-w}) = w$, which is seen to satisfy $0 < (1 - e^{-w}) = w < 1$ for all $w > 0$, it follows that $p_h(s) \in (0; 1)$. Next let

$$e_0(s) = \frac{\frac{3}{4} p_h(s) s}{1 - \frac{3}{4} + \frac{3}{4} p_h(s) s} \in (0; 1) \quad (\text{A.37})$$

[so that (3.26) holds] and let $V_0(s)$ and $V_1(s)$ be defined in terms of (3.8) and (3.9) by

$$V_0(s) = \frac{1 - e_0(s)}{1 - e_0(s) e_1} \frac{\bar{A} U_0(s)}{1 - \frac{3}{4}} + \frac{e_0(s) (1 - e_1)}{1 - e_0(s) e_1} \frac{\mu U_1}{1 - \frac{3}{4}} > 0 \quad (\text{A.38})$$

$$V_1(s) = \frac{1 - e_1}{1 - e_0(s) e_1} \frac{\mu U_1}{1 - \frac{3}{4}} + \frac{e_1 (1 - e_0(s))}{1 - e_0(s) e_1} \frac{\bar{A} U_0(s)}{1 - \frac{3}{4}} > 0 \quad (\text{A.39})$$

so that (3.27) and (3.28) must also hold [by the definitions of (3.8) and (3.9)]. Hence all values $[u(s); v(s); p_h(s); e_0(s); V_0(s); V_1(s)]$ are uniquely defined, have the correct domains, and satisfy conditions (3.23) through (3.28). To see that the positivity condition (3.29) also holds, observe that since the relations (A.24)

through (A.26) are independent of (3.29), it follows that these relations must continue to hold, so that

$$\frac{3}{4} p_h(s) s [V_1(s) \text{ ; } V_0(s)] = \frac{\bar{A} e_0(s) (1 \text{ ; } e_1)}{1 \text{ ; } e_0(s) e_1} [U_1 \text{ ; } U_0(s)] \text{ ;} \quad (\text{A.40})$$

Finally, since the relations in (A.28) through (A.30) are also independent of (3.29), it follows from the definition of $W(s)$ that

$$\begin{aligned} \frac{1}{s} \frac{U_1 \text{ ; } U_0(s)}{U_0(s)} &= 1 + W(s) = 1 + \frac{u}{1 \text{ ; } u} \\ &= \frac{1 \text{ ; } e_0(s) e_1}{e_0(s) (1 \text{ ; } e_1)} \\ \frac{\bar{A} e_0(s) (1 \text{ ; } e_1)}{1 \text{ ; } e_0(s) e_1} [U_1 \text{ ; } U_0(s)] &= \frac{s}{1 \text{ ; } s} U_0(s) \\ &= s (1 \text{ ; } s)^{-1} b \text{ ;} \end{aligned} \quad (\text{A.41})$$

which together with (A.40) [and the positivity of s] is seen to imply that (3.29) must hold. Thus the vector $[s; u(s); v(s); p_h(s); e_0(s); V_0(s); V_1(s)]$ constitutes the unique endogenous-search equilibrium for these parameters, and the result is established. ■

A.4. Proof of Lemmas A.1 and A.2

If the function g is defined for all $s \in (0; 1]$ by

$$g(s) = G \frac{W(s)}{1 + W(s)} \text{ ; } s \quad (\text{A.42})$$

then to complete the proof of Theorem 3.2 it remains to be shown that g has a unique root in the open interval $(0; s_a)$. To do so, we first show that if

$$u(s) = \frac{W(s)}{1 + W(s)} \quad (\text{A.43})$$

$$v(s) = u(s) + a \quad (\text{A.44})$$

for each $s \in (0; 1]$, then

Lemma A.1. For each $a > \frac{1}{2}$ there exists a unique search intensity, $s_a \in (0; 1)$, such that

$$\min_{s \in (0; s_a)} f(u(s); v(s)) g > 0 \text{ , } \quad s \in (0; s_a) \quad (\text{A.45})$$

Proof. Observe first that if $a \geq 0$, then by (A.43) and (A.44)

$$\min_{f(u); v(s)} g = u(s) > 0, \quad W(s) > 0: \quad (\text{A.46})$$

On the other hand, if $a < 0$, then

$$\begin{aligned} \min_{f(u); v(s)} g = v(s) > 0, \quad u(s) + a > 0 \\ & , \quad \frac{W(s)}{1 + W(s)} + a > 0 \\ & , \quad W(s) > \frac{a}{1 + a} \end{aligned} \quad (\text{A.47})$$

[where $1 + a > 0$ by hypothesis]. Hence $\min_{f(u); v(s)} g > 0$ implies

$$W(s) > \max \left\{ 0; \frac{a}{1 + a} \right\} \quad (\text{A.48})$$

But since (A.32) implies that

$$\lim_{s \rightarrow 0} W(s) = \lim_{s \rightarrow 0} \frac{1 - (1 - s)^{1/b}}{s} \frac{U_1 - (1 - s)^{1/b} U_0}{(1 - s)^{1/b}} \Big|_{s=1} = 1 \quad (\text{A.49})$$

it follows that (A.48) always holds for values of s near zero. Moreover, since (A.32) also implies that

$$\begin{aligned} W(s) > 0, \quad (1 - s)[U_1 - U_0(s)] > s U_0(s) \\ & , \quad (1 - s)^{1/b} U_1 > (1 - s + s)^{1/b} U_0(s) \end{aligned} \quad (\text{A.50})$$

we see from the positivity of W implies that $W(1) < 0$. Hence in the last inequality of (A.50), the strict concavity of the left-hand side and linearity of the right-hand side are easily seen to imply that these two terms must be equal at exactly one intermediate point, $s_0 \in (0; 1)$, and that (A.50) holds for $s \in (0; s_0)$. Next observe that

$$W'(s) = \frac{1}{s^2} \left[1 + \frac{U_1}{U_0(s)} (s - 1) \right] \quad (\text{A.51})$$

implies

$$W'(s) < 0, \quad (1 - s) U_1 > U_0(s); \quad (\text{A.52})$$

Moreover, since

$$\begin{aligned} & (1 - s)[U_1 - U_0(s)] > s U_0(s) \\ &) \quad (1 - s) U_1 > (1 - s + s) U_0(s) \\ &) \quad (1 - s) U_1 > U_0(s) \end{aligned} \quad (\text{A.53})$$

[where the last line follows from the inequality, $(1 - \alpha s)(1 - \alpha s + \alpha s) > 1 - \alpha s$], we see from (A.50) through (A.53) that $W(s) > 0 \implies W'(s) < 0$. Hence W is strictly decreasing on the interval $(0; s_0)$, and it may be concluded that for each $\alpha > 1$ there is a unique $s_\alpha \in (0; s_0]$ satisfying $W(s_\alpha) = \max_{s \in (0; s_0]} W(s) = 0$. Finally, since the decreasing monotonicity of W also implies that (A.48) holds in $s \in (0; s_\alpha)$, we see that s_α satisfies (A.45). ■

Given this admissible range on s , our main result is the following:

Lemma A.2. For each $\alpha > 1$, the function g has a unique root in the open interval $(0; s_\alpha)$.

Proof. To verify the existence of such root, we first show that it suffices to establish the following four properties of the function g :

$$g(0) = 0 \tag{A.54}$$

$$g'(0) < 0 \tag{A.55}$$

$$\lim_{s \rightarrow s_\alpha} g(s) > 0 \tag{A.56}$$

$$\forall s \in (0; s_\alpha); g'(s) = 0 \implies g''(s) > 0 \tag{A.57}$$

For it will then follow from (A.54) and (A.55) that $g(s) < 0$ near $s = 0$, and from (A.56) that $g(s) > 0$ near $s = s_\alpha$. Hence by continuity, g must pass upward through zero at some intermediate point, $s^* \in (0; s_\alpha)$, so that in particular, $g'(s^*) = 0$. To see that this root must be unique, observe first that since (A.54) and (A.55) also imply that g is negative on a maximal open interval $(0; s_m)$, it may be assumed that $s^* = s_m$, i.e., that s^* is the smallest root of g in $(0; s_\alpha)$. Next observe that since g is continuously differentiable in $(0; s_\alpha)$, it follows from (A.57) that

$$g'(s^*) > 0 \tag{A.58}$$

[for if $g'(s^*) = 0$, then $g''(s^*) > 0$ would imply that $g'(s) < 0$ holds for every s in some open interval $(s^* - \epsilon; s^*)$, and hence that g does not pass upward through zero at s^*]. Hence if g were to have an additional root, $s_\alpha \in (s^*; s_\alpha)$, then (A.58) [together with the continuity of g] would imply that g must achieve a (differentiable) maximum at some interior point $\xi \in (s^*; s_\alpha)$. Finally, since the maximality conditions $g'(\xi) = 0$ and $g''(\xi) < 0$ would then contradict (A.57), it follows that no such root can exist, and hence that s^* is unique. To establish conditions (A.54) through (A.57), we proceed in order:

Proof of (A.54). Observe from (A.43) and (A.49) that

$$\lim_{s \rightarrow 0} W(s) = 1 \implies \lim_{s \rightarrow 0} u(s) = 1$$

and hence that

$$\lim_{s \downarrow 0} g(s) = G[1; 0] = \frac{1}{2} \left(1 - \frac{1}{2} \right) (1 + a)^{-1} e^0 = 0 \quad (\text{A.59})$$

Thus it follows from the continuity of g that (A.54) must hold.

Proof of (A.55). To compute the derivative

$$g'(s) = \frac{\partial G}{\partial u} u'(s) + \frac{\partial G}{\partial s} \quad (\text{A.60})$$

observe first from (A.43) and (A.51) that

$$u'(s) = \frac{W'(s)}{[1 + W(s)]^2} = \frac{\frac{1}{s} \left(1 + \frac{U_1}{U_2(s)} \right)}{\left[1 + \frac{1}{s} \left(1 + \frac{U_1 (1-s)^b}{(1-s)^b} \right) \right]^2} \quad (\text{A.61})$$

Hence $U_0(0) = b$ implies that

$$\lim_{s \downarrow 0} u'(s) = \frac{\frac{1}{s} \left(1 + \frac{U_1}{b} \right)}{\left[1 + \frac{1}{s} \left(1 + \frac{U_1}{b} \right) \right]^2} = \frac{\bar{A} b}{U_1 + b} \quad (\text{A.62})$$

Also from (A.18) and (A.22) we see that

$$\frac{\partial G}{\partial u} [u(0); 0] = \left(1 - \frac{1}{2} \right) e^0 \frac{1 + a \cdot 0}{1 + a} = \frac{1}{2} \quad (\text{A.63})$$

$$\frac{\partial G}{\partial s} [u(0); 0] = \left(1 - \frac{1}{2} \right) e^0 = \frac{1}{2} \quad (\text{A.64})$$

Thus, by substituting (A.62) through (A.64) into (A.60), we see that

$$g'(0) = \frac{1}{2} \left(\frac{\bar{A} b}{U_1 + b} + \frac{1}{2} \right) \quad (\text{A.65})$$

which together with (A.31) implies that

$$g'(0) < 0, \quad \frac{\bar{A} b}{U_1 + b} < \left(1 - \frac{1}{2} \right) \\ , \quad U_1 + b > \frac{\bar{A}}{3/4} \left(1 - \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{2} \right) b \quad (\text{A.66})$$

But by the positivity condition (3.22) it follows that

$$U_1 + b > \frac{\bar{A}}{3/4} \frac{1 - \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{2}}{1 - \frac{1}{2}} b > \frac{\bar{A}}{3/4} \left(1 - \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{2} \right) b \quad (\text{A.67})$$

and thus that (A.55) must hold.

Proof of (A.56). Suppose first that $a \geq 0$ (so that by definition $s_a = s_0$), and observe from (A.32) and (A.43) that

$$\lim_{s \rightarrow s_0} W(s) = 0 \implies \lim_{s \rightarrow s_0} u(s) = 0 \quad (\text{A.68})$$

Hence from (A.42) together with (A.8) we see that

$$\lim_{s \rightarrow s_0} g(s) = \frac{1}{2} (1 + \frac{1}{2}) a^{-1} e^0 = \frac{1}{2} > 0 \quad (\text{A.69})$$

Next suppose that $a < 0$. In this case, (A.32), (A.43), and (A.47) imply that

$$\begin{aligned} \lim_{s \rightarrow s_a} W(s) &= \frac{a}{1+a} \\ \implies \lim_{s \rightarrow s_a} u(s) &= u(s_a) = a \\ \implies \lim_{s \rightarrow s_a} v(s) &= 0; \end{aligned} \quad (\text{A.70})$$

which together with $1 + a > 0$; allows us to conclude that

$$\begin{aligned} \lim_{s \rightarrow s_a} g(s) &= \frac{1}{2} (1+a) (1 + \frac{1}{2}) (0)^{-1} e^{1-a} \\ &= \frac{1}{2} (1+a) > 0 \end{aligned} \quad (\text{A.71})$$

Hence we see that (A.56) holds in all cases, and it remains only to establish (A.57).

Proof of (A.57). To establish this key property, we first introduce the following simplifying notation. For all $s \in (0; s_a)$ let

$$L(s) = \frac{u(s)}{u(s) + a} = \frac{W(s)}{a(1+W(s)) + W(s)} = \frac{W(s)}{(1+a)W(s) + a} \quad (\text{A.72})$$

and [recalling (A.18) and (A.22)] let

$$M(s) \stackrel{\partial G}{\partial u} [u(s); s] = (1 + \frac{1}{2}) e^{i \circ s L(s)} [1 + \circ s + \circ s L(s)] \quad (\text{A.73})$$

$$Q(s) \stackrel{\partial G}{\partial s} [u(s); s] = (1 + \frac{1}{2}) \circ u(s) e^{i \circ s L(s)} \quad (\text{A.74})$$

With this notation, it follows (by dropping functional dependencies on s) that

$$g^0 = M u^0 + Q \quad (\text{A.75})$$

and hence that

$$g^{00} = M u^{00} + M^0 u^0 + Q^0 \quad (\text{A.76})$$

To evaluate the first term of (A.76), observe first from (A.75) together with (A.61) and (A.74) that $g^0 = 0$ is

$$M = i Q = u^0 = \frac{(1+W)^2}{1+W^0} (1-i/2) e^{i s L} \frac{\bar{A}}{1+W} e^{i s L} \quad (\text{A.77})$$

Next observe from (A.61) that

$$u^{00} = \frac{d}{ds} \frac{1+W^0}{(1+W)^2} = \frac{1}{(1+W)^2} \bar{A} W^0 i \frac{2(W^0)^2}{1+W} \quad (\text{A.78})$$

and hence (after some reduction) that

$$M u^{00} = (1-i/2) e^{i s L} \frac{\bar{A}}{1+W} \frac{W^0}{W^0} i \frac{2W^0}{1+W} \quad (\text{A.79})$$

To evaluate the second term of (A.76), observe from (A.73) that

$$M^0 = i (1-i/2) e^{i s L} [1-i s (1-L) (L+sL^0)] \quad (\text{A.80})$$

which together with (A.61) yields

$$M^0 u^0 = i (1-i/2) e^{i s L} [1-i s L (1-L) (L+sL^0)] \frac{(1+W^0)}{(1+W)^2} \quad (\text{A.81})$$

To evaluate the last term of (A.76), observe first from (A.74) that

$$\begin{aligned} Q^0 &= i (1-i/2) e^{i s L} \frac{d}{ds} \frac{W}{1+W} e^{i s L} \\ &= i (1-i/2) e^{i s L} \frac{1+W^0}{(1+W)^2} i \frac{\bar{A}}{1+W} (L+sL^0) \end{aligned}$$

But since (A.72) implies

$$L^0 = \frac{d}{ds} \frac{W}{(1+a)W+a^1} = \frac{a^1 W^0}{[(1+a)W+a^1]^2}; \quad (\text{A.82})$$

it follows (after some reduction) that

$$Q^0 = i (1-i/2) e^{i s L} \frac{(1+W^0)}{(1+W)^2} [1-i s L (1-L)] i \frac{\bar{A}}{1+W} L \quad (\text{A.83})$$

Hence, combining (A.79), (A.81), (A.83), factoring out the (positive) common term, $(1 - \frac{1}{2}) e^{i s L} = (1 + W)$, and simplifying, it follows from (A.76) that when $g^0 = 0$, we have $g^{00} > 0$ i.e.

$$0 < W \frac{W^{00}}{W^0} + \frac{2W^0}{1+W} + W L \frac{1}{1+W} \frac{1}{2 [1 - \frac{1}{2} s L (1 - L)]} s^2 (1 - L) L^0 \quad (\text{A.84})$$

To reduce (A.84) further, we next use (A.51) together with the identity, $U_0^0 = U_0 = (1 - s)$, to evaluate

$$W^{00} = \frac{d}{ds} \frac{1}{s^2} \left(1 + \frac{U_1}{U_0} (s - 1) \right) = \frac{1}{s^3} \left(2 + \frac{U_1}{U_0} \right) + \frac{1}{1 - s} \frac{1}{s^2} \quad (\text{A.85})$$

Then, after some manipulation, the ratio, $W^{00} = W^0$, reduces to

$$\frac{W^{00}}{W^0} = \frac{s U_1 (1 - s)}{(1 - s) [(1 - s) U_1 - U_0]} = \frac{2}{s} \quad (\text{A.86})$$

Upon substituting (A.86) into (A.84), dividing through by $2W = s > 0$, and rearranging terms, we obtain the equivalent condition

$$1 < \frac{W^0}{W} s \frac{W + 1 [1 - \frac{1}{2} s L (1 - L)]}{(1 + W) \frac{W^0}{W} \frac{L^0}{L} s^2 (1 - L) + 1} + \frac{1}{2} \frac{s}{1 - s} \frac{U_1 s (1 - s)}{U_1 (1 - s) - U_0} \quad (\text{A.87})$$

To simplify this condition further, recall from (A.53) that $(1 - s) U_1 > U_0(s)$ for all $s \in (0; s_a)$, and hence that the last term in (A.87) is always positive. Hence it suffices to show that

$$1 < \frac{W^0}{W} s \frac{W + 1 [1 - \frac{1}{2} s L (1 - L)]}{(1 + W) \frac{W^0}{W} \frac{L^0}{L} s^2 (1 - L) + 1} \quad (\text{A.88})$$

The second term in (A.88) can be simplified by observing from (A.72) that

$$L^0 = \frac{a^2 W^0}{[(1 + a)W + a]^2} \quad \Rightarrow \quad \frac{L^0}{L} = \frac{a^2}{(1 + a)W + a} \frac{W^0}{W} \quad (\text{A.89})$$

and in addition, from (A.32) and (A.51) that

$$\frac{W^0}{W} = \frac{\frac{1}{s} \frac{U_1}{U_0} (1-s)}{1 + \frac{1}{s} \frac{U_1}{U_0} (1-s)} = \frac{\frac{1}{s} [U_0 + U_1 (1-s)]}{(1-s)(U_1 + U_0) + sU_0} \quad (\text{A.90})$$

By using these results, and letting

$$R = \frac{W^0}{W} = \frac{(1-s) U_1 + U_0}{(1-s)(U_1 + U_0) + sU_0}; \quad (\text{A.91})$$

we can rewrite the right hand side of (A.88) as

$$\begin{aligned} R & \frac{(W+1)^{1-s} L (1-L)^s}{2} + \frac{sL}{2} \frac{1}{1+W} R^2 \frac{a^1}{(1+a)W+a^1} (1-L)^s + \\ & = R \frac{1-s}{1+W} L (1-L)^s + \frac{sL}{2} \frac{1}{1+W} R^2 \frac{a^1}{(1+a)W+a^1} (1-L)^s + \frac{sL}{2} \end{aligned} \quad (\text{A.92})$$

so that condition (A.88) becomes

$$1 < R \frac{1-s}{1+W} L (1-L)^s + \frac{sL}{2} \frac{1}{1+W} R^2 \frac{a^1}{(1+a)W+a^1} (1-L)^s + \frac{sL}{2} \quad (\text{A.93})$$

We next show that $R(s) > 1$ for all $s \in (0; s_a)$. To do so, recall from (A.53) that $(1-s) U_1 + U_0 > 0$, and hence [again from the inequality $(1-s)(1-s+s) > 1-s$ and positivity of $(1-s+s)$] that

$$\begin{aligned} (1-s) U_1 + U_0 & > (1-s+s) [(1-s) U_1 + U_0] \\ & > (1-s) U_1 + (1-s+s) U_0 \\ & = (1-s)(U_1 + U_0) + sU_0 \end{aligned} \quad (\text{A.94})$$

Thus by (A.91) we see that $R > 1$ on $(0; s_a)$. To use this result, we next show that both the terms in on the right hand side of (A.93) involving R are always nonnegative. To establish nonnegativity of the first term, observe first that if $a \geq 0$, then by (A.72) it follows that $L(s) \in [0; 1]$, and hence that $L(1-L) \geq 0$.

[0; 1=4]. This together with $s^\circ [1=(1+W)] \geq 2(0; 1)$ implies that the bracketed expression in the ...rst term is positive. Next, if $a < 0$, observe again from (A.72) that $L(s) > 1$, and hence that $L(1; L) < 0$, so that the bracketed expression is again positive. Hence $R > 1$ implies that the ...rst term is in fact positive. Turning to the second term, we need only consider the product, $A \zeta(1; L)$, with $A = a^{-1}[(1+a)W + a^{-1}]$. Again if $a \geq 0$, then both A and $(1; L)$ are nonnegative, so that their product is also. Finally, if $a < 0$, then since the definition of s_a implies from (A.48) that $W(s) > a^{-1}(1; a)$ and hence that $(1+a)W + a^{-1} > 0$, it now follows that both A and $(1; L)$ are negative, and again have a nonnegative product. Given these nonnegativity properties, it thus suffices to establish (A.93) with R replaced by one, which (after regrouping terms) is equivalent to showing that

$$1 < 1; s^\circ \frac{\bar{A}^{-1}}{1+W} L(1; L) \leq 1; \frac{1}{2} \frac{\bar{A}^{-1} a^{-1}}{(1+a)W + a^{-1}} + \frac{sL}{2} \quad (\text{A.95})$$

By subtracting one, then dividing by (sL) , and using (A.72), this reduces to:

$$\begin{aligned} \frac{1}{2} &> \frac{\bar{A}^{-1}}{1+W} L(1; L) \leq 1; \frac{1}{2} \frac{\bar{A}^{-1} a^{-1}}{(1+a)W + a^{-1}} \\ &= \frac{\bar{A}^{-1} a^{-1}}{(1+a)W + a^{-1}} \leq 1; \frac{1}{2} \frac{\bar{A}^{-1} a^{-1}}{(1+a)W + a^{-1}} \end{aligned} \quad (\text{A.96})$$

Finally, letting $x = a^{-1}[(1+a)W + a^{-1}] < 1$, and observing that

$$\begin{aligned} (1-x)^2 &= 1 - 2x + x^2 > 0 \\ &\Rightarrow 1 > 2x - x^2 \\ &\Rightarrow \frac{1}{2} > x - \frac{1}{2}x \end{aligned} \quad (\text{A.97})$$

we may conclude that (A.96) holds, and hence that the result is established. ■