

Bargaining Power in Communication Networks*

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Abstract

The aim of this paper is to determine how the place of a player in a network of communications affects her bargaining power with respect to the others. We adapt the Rubinstein-Ståhl two-player noncooperative bargaining game of alternating offers to the case of n players connected through a graph. We show that this game has a unique stationary subgame perfect equilibrium outcome from which we derive a bargaining power measure. This bargaining power measure satisfies properties of efficiency, anonymity, monotonicity, local impact, weighted fairness and fair reallocation that we define and discuss.

Keywords: communication, network, graph, noncooperative bargaining, bargaining power.

Journal of Economic Literature Classification Numbers: C72, C78, D20

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1. Introduction

In many social and economic situations, agents are in direct contact with only a limited subset of other agents. When this is the case, bilateral voluntary meetings are limited to those pairs of agents that can communicate with each other directly or that know each other directly. The network of communications plays then a major role when determining the aggregate behavior of the group. In standard economic analysis, though, the nature of the communication structure supporting agent-to-agent meetings or talks is rarely made explicit and aggregate behavior is commonly inferred from studying the behavior of one representative agent in isolation.¹

The aim of this paper is to determine how the place of a player in a network of communications affects her bargaining power with respect to the others. To this purpose, we take explicit account of the communication network that we model by an undirected graph. The nodes represent the players and a link between two nodes means that the corresponding players can communicate directly or, equivalently, that they are direct neighbors. We explore the implications of the interaction system constituted by such a communication structure in a simple bargaining context.

The bargaining game we consider is an adaptation of the Rubinstein-Stähl alternating offers game [see Rubinstein (1982)]. An informal description is as follows. Some player (the promoter) has an idea about a new and profitable economic activity and she requires the service of an associate to setup the firm that exploits the gains from her idea. Alternatively, one can think of some player being aware of a public authority subsidizing the formation of joint ventures (like the antitrust authority in the case of procompetitive collaborative agreements between competitors) or research alliances (like science foundations of any kind). More generally, we have in mind situations where pairing members creates value which must be divided between two individuals. The promoter then randomly selects a partner among her set of neighbors (acquaintances) and makes her a splitting offer. This bargaining partner can either accept the offer or reject it. In case of rejection, the respondent becomes the new proposer at the next round and her bargaining partner is again selected randomly among her neighbors. This

¹Exceptions are Furth (1998), Greenberg and Weber (1983) and Kirman *et al.* (1986) that explicitly focus on the role of communication in exchange markets.

noncooperative bargaining game ends up when, and if, an agreement is reached between two neighboring negotiation partners.

The three main features of the model described above are the following. First, players are only allowed to make proposals to their direct neighbors. The communication network introduces natural restrictions over feasible pairwise meetings and only players that are in direct contact with each other can negotiate together. In that sense, our model is one of local strategic interaction.² Second, bargaining is sequential and any player can only bargain with one partner at a time. In particular, simultaneous offers to two different neighbors are not possible. Finally, the pairs of neighbors that bargain at every round are chosen at random within the graph constraints and activated from neighborhood to neighborhood. This random matching process implicitly assumes some sort of boundedly rational behavior from the part of the agents during the bargaining phase and gives an evolutionary flavor to our model. There is a sense in which agents are boundedly rational because we do not consider optimal bargaining partner selection as a function of players' relative positions in the network of communications. Co-bargainer selection is not considered here as a strategic issue. We believe, though, that these assumptions make sense in our framework where the goal is to evaluate the impact of a graph with a cooperative concept. It is indeed a common feature of cooperative game theory not to necessarily make explicit the whole decision process underlying individual actions. The present paper thus owes a lot, at least in spirit, to the recent evolutionary literature on local interaction with seminal contributions by Foster and Young (1990), Kandori *et al.* (1993) and Young (1993). Consequently, this work cannot be read as a strict contribution to the Nash program.³

We first show that the n -player noncooperative bargaining game on the network of communications has a unique stationary subgame perfect equilibrium. We then determine how ex post payoffs of this game, corresponding to the unique equilibrium agreed-upon shares, depend on the underlying communication structure supporting players' talks. In particular, we find that the equilibrium splits are independent of the network and tend to be egalitarian when the agents are patient

²Ideally though a model should contain both local interactions (e.g. behavioral influences) and global interactions (e.g. market or social conditions), but this is a subject for future research.

³The evolutionary approach, though, has already proved to be fruitful for the analysis of bargaining models. For instance, Young (1993) recovers (and generalizes) the Nash solution as the unique stable outcome of an evolutionary version of the Nash demand game. Sáez-Martí and Weibull (1999) analyze the robustness of these results to variations of the individual boundedly rational behavior.

and have equal discount factors. Ex post payoffs do not depend on the particular shape of the network when the population is homogeneous in time preferences. The intuition for this result relies on the Outside Option Principle [Binmore *et al.* (1989)]. In our game, opting out (switching over to an alternative partner) is payoff-equivalent to playing a standard two-player alternating offers game. Therefore, the outside option constituted by contacting an alternative available partner is a non-credible threat and the bargaining outcome is immune to its effect. Still, the communication structure plays a central role regarding expected payoffs. Indeed, a well connected agent has more possibilities to be chosen as the bargaining partner of some other agents.⁴

Ex ante payoffs given by the expected equilibrium partition of our bargaining game with random selection of the negotiators define an allocation rule that we fully characterize. By removing the friction introduced in the model by the discount factor (cost to delay through disagreement) we then obtain a measure of the players' bargaining power as a function of their relative positions in the network of communications. This bargaining power measure determines how the place of a player in the network affects her bargaining strength relative to the others, and thus captures the asymmetries induced by the communication structure that restricts pairwise meetings.

The first paper to explicitly model a situation where players have restricted communication possibilities is Myerson (1977). Since then, communication restrictions have received great attention in cooperative game theory. Contributions on graph-restricted cooperative games include Borm *et al.* (1992), Grofman and Owen (1982), Hamiache (1999), Myerson (1980), Owen (1986), Rosenthal (1988a, 1988b, 1992) and Vásquez-Brage *et al.* (1996). This line of research focuses on the effect of communication linkages on resource allocation and on coalitional values. Borm *et al.* (1994) provide an exhaustive survey of this literature. The question of endogenous formation, stability and efficiency both for directed and undirected communication networks has also been extensively analyzed. Examples of this strand of research are Aumann and Myerson (1988), Bala and Goyal (1998, 1999), Dutta and Mutuswami (1997), Dutta *et al.* (1998), Goyal (1993), Jackson (1999), Jackson and Watts (1998) and Jackson and Wolinsky (1996). Finally, communication networks have also been widely studied from a sociological perspective [see Wasserman and Faust (1994) for a survey]. Among other issues, the problem of power distribution is of central concern. Skvoretz and Willer (1993)

⁴The communication structure is also very important for ex post payoffs when players have different discount factors.

provide a quick overview of recent theories on this subject.

The paper is organized as follows. Section 2 presents the general model of n -player bargaining on the communication structure and states the main result of uniqueness of the subgame perfect equilibrium outcome. The resulting allocation rule and the bargaining power measure in the communication network are defined in Section 3. The main properties of this measure are presented in Section 4. An appendix contains all the proofs.

2. Bargaining and communication networks

2.1. The communication network

Let $N = \{1, \dots, n\}$ be a finite set of players ($n \geq 2$) with time preferences represented by a discount factor $\delta_i \in (0, 1)$ for all $i \in N$. A communication network g on N is a set of pairwise communication channels, where the communication channels are undirected bilateral links between players. A graph on N is a set of unordered pairs of distinct members of N . We can therefore model communication networks by graphs: the nodes are identified with the players and the links are identified with the communication channels. A link in g between two players i and j , denoted by $\{i : j\}$, means that these players can communicate directly or, equivalently, that they are direct neighbors.⁵ The complete graph constituted by all the possible links between two players in the population is thus the set of all subsets of N of size 2. We denote it by g^N . The set of all possible graphs on N is then $\mathcal{G} = \{g \mid g \subseteq g^N\}$. We next introduce a definition:

Definition 1. For all $g \in \mathcal{G}$ and for all $i \in N$, i 's neighborhood denoted by $N(g, i)$ is the set of players in N that communicate directly with i when the communication structure is g that is, $N(g, i) = \{j \in N \setminus \{i\} \mid \{i : j\} \in g\}$.

A similar type of neighborhood structure can be found for instance in Bala and Goyal (1998). Note that by definition $i \notin N(g, i)$, and $i \in N(g, j)$ is equivalent to $j \in N(g, i)$. In words, the neighborhood relationship is non reflexive and symmetric. We denote by $n(g, i)$ the cardinality of $N(g, i)$.⁶ To avoid trivialities

⁵We will use the terms *communication network*, *communication structure*, *graph* or *network* without distinction.

⁶In graph theory $n(g, i)$ is the degree of node i , that is the number of nodes adjacent to i or, equivalently, the number of links that are incident with i .

we restrict from now on to $\mathcal{G}^* = \{g \in \mathcal{G} \mid n(g, i) \geq 1, \forall i \in N\}$, the set of communication networks where all players have at least one neighbor, implying that no player is totally isolated.

2.2. The noncooperative bargaining game

An informal version of the game we consider is the following. Suppose that a player has an idea about a new and profitable economic activity to exploit jointly with an associate. The set of potential partners the promoter has at her disposal, presumably limited, is constituted by the players in the population she is in direct contact with. The promoter contacts one player among her acquaintances, proposes this player to become her associate, and engages a bilateral negotiation to fix their respective shares of the outcome delivered by the new business activity. If the two players agree on how to share the benefits the negotiation ends, the firm is created, and its profits are distributed according to the agreed shares. Of course, these shares depend on the players preferences, on the rules of negotiation and on the underlying communication structure supporting players' talks.

More formally, given a communication network $g \in \mathcal{G}^*$, the sequential noncooperative game we analyze is described as follows:

In each round there are two active players, a proposer $i \in N$ and a respondent $j \in N(g, i)$ who is in the neighborhood of the proposer. In the first round, one player i is chosen at random out of N to be the initial proposer, with all players being equally likely to be selected (probability $1/n$). A respondent j is then chosen at random among the proposer's neighbors, with all players in the neighborhood of i being equally likely to be selected (probability $1/n(g, i)$). The proposer makes a two-player/one-cake proposal to the respondent. If the respondent accepts the offer, the game ends with these payoffs. If she rejects the offer, we move to the next round where the respondent j becomes the new proposer and a new respondent k is now chosen at random among the proposer's neighbors $N(g, j)$, with all players in the neighborhood of j being equally likely to be selected (probability $1/n(g, j)$). And so on. The game proceeds until, and if, a bilateral agreement is reached.

According to this bargaining procedure, a player can only make offers to her neighbors and can never make simultaneous offers to two different neighbors. Also,

once a proposer is chosen, the respondent (co-bargainer) is selected randomly from her set of neighbors. The hypothesis of random selection may seem arbitrary at a first sight, but the equilibrium agreed-upon partition between two neighbors when players have equal discount factors is always half-half irrespective of their location in the graph (either well-connected or bad-connected). Therefore, no promoter has an advantage to discriminate among potential partners, and it is reasonable to assume that all neighbors are treated on an equal footing.⁷ In case of rejection, the respondent becomes the new proposer at the next round and any of the rejector's neighbors can become her bargaining partner. In particular, the new respondent can differ from the neighbor she's just been negotiating with in the previous round. Therefore, bargaining partners (pairs of players) are sequentially activated from neighborhood to neighborhood and bargaining proposals do not necessarily consist in a sequence of offers and counter-offers as in the standard two-player noncooperative bargaining game. This possibility of switching over to an outside neighbor may influence the equilibrium of the game.

Bargaining with many partners may proceed in a variety of ways. Although the outcome is almost always dependent on the framework within which the negotiation takes place, different models of n -player bargaining coexist in the literature.⁸ Our bargaining procedure is just a variation of the Rubinstein-Ståhl alternating offers game adapted to the case of n players connected through a graph. In the presence of communication networks, bilateral volunteer meetings are limited to those pairs of players that can communicate with each other directly, or that know each other directly. Our model takes this restrictions into account very naturally by allowing only direct neighbors to enter into bilateral negotiations.

⁷Also, any deterministic selection rule fixed a priori (e.g. selecting the neighbor with smaller neighborhood size, etc.) does not change the result when the population is homogeneous in time preferences: ex post payoffs are always distributed according to a standard half-half splitting rule. On the contrary, when the population is heterogeneous in time preferences, players have incentives to select their partners strategically according to their position in the network [see Calvó-Armengol (1999)].

⁸Indeed, models of noncooperative bargaining among n participants are either applied to the case of a pie of fixed size [see for instance Calvó-Armengol (1999), Chae (1993), Chae and Yang (1994), Haller (1986), Herrero (1985), Jun (1987), Krishna and Serrano (1996)], or to games in coalitional form [as for instance in Gul (1989), Hart and Mas-Colell (1996), Okada (1996)]. Bargaining procedures employed are such that players meetings are either bilateral [Calvó-Armengol, Chae and Yang, Jun, Gul], or multilateral [Krishna and Serrano, Haller, Hart and Mas-Colell, Herrero, Okada]. Also, responses to offers are either sequential [Hart and Mas-Colell, Okada], or simultaneous [Haller, Krishna and Serrano].

2.3. Analysis of the game

We concentrate on *stationary subgame perfect equilibria* that is, on those subgame perfect equilibria where strategies only depend on the identity of the players that bargain, but neither on history nor on the current round. We first need to introduce some notations.

Definition 2. For all $g \in \mathcal{G}^*$ and for all $i \in N$ and $j \in N(g, i)$, $(a_{ij}(g), 1 - a_{ij}(g))$ is the two-player/one-cake proposal made by player i to her neighbor j .

If the respondent j accepts such a proposal she gets $1 - a_{ij}(g)$ while the proposer i ends up with $a_{ij}(g)$. Note that players can only make proposals to their direct neighbors, the only players with whom they can communicate directly. The communication network can thus be seen as a network of bargaining possibilities. We assume from now on and without loss of generality that players bargain for a cake of size one. The main result of this section is:

Proposition 1. For all $g \in \mathcal{G}^*$, the n -player bargaining game on g has a unique stationary subgame perfect equilibrium. For all $i \in N$ and $j \in N(g, i)$ the equilibrium shares are characterized by:

$$1 - a_{ij}(g) = \delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} a_{jk}(g) \quad (2.1)$$

Equations (2.1) mean that the proposer i concedes to any of her neighbors $j \in N(g, i)$ the expected payoff this neighbor can get in the continuation game if she rejects i 's proposal.⁹ In a stationary subgame perfect equilibrium a player agrees to a proposal if it offers at least as much as what she can expect to get in the future appropriately discounted. At equilibrium, players are indifferent between their share as a respondent and their share as a delayed proposer. Note that we do not need to assume that players have a common discount factor to obtain uniqueness which holds for any distribution of discount factors $\delta_i \in (0, 1)$.

When the players have a common discount factor δ (population homogeneous in time preferences), the stationary subgame perfect equilibrium shares take a particularly simple form:

⁹These equations define a system of $\sum_{i \in N} n(g, i)$ equations with $\sum_{i \in N} n(g, i)$ unknowns $[a_{ij}(g)]_{i \in N, j \in N(g, i)}$.

Corollary 1. *When the population is homogeneous in time preferences with common discount factor $\delta \in (0, 1)$, at equilibrium all players make the standard cake division proposal $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ to their neighbors.*

In words, whatever the location of a player (either isolated or well-connected) and whatever the bargaining partner selected in her neighborhood (either isolated or well-connected) the first player always proposes in equilibrium the standard partition $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ and her partner immediately accepts it. At the limit $\delta \rightarrow 1$ we recover the standard $(\frac{1}{2}, \frac{1}{2})$ partition of the Rubinstein-Stähl two-player game. Hence, when two neighbors agree on a joint collaboration in a project the derived benefits are always distributed according to a standard half-half splitting rule irrespective of their eventual asymmetric locations in the graph. Whoever the promoter, all her neighbors are worth *ex post* the same as potential associates as long as the discount factor is common to all players in the population. No promoter has thus incentives to discriminate among her neighbors and it is reasonable to assume that all neighbors are treated on an equal footing, the bargaining partner being selected with uniform probability within the neighborhood.¹⁰ The intuition for this result is the following. The population is homogeneous in time preferences and time preferences are the leading force governing players' behavior during negotiations. Opting out -switching over to an alternative partner- is payoff-equivalent to playing a standard two-player alternating offers game. Therefore, outside options do not constitute a credible threat and neighborhood sizes have no effect on the *ex post* equilibrium partition [Shaked *et al.* (1989) and the Outside Option Principle].

3. The bargaining power measure

3.1. The allocation rule resulting from the bargaining game

Given a communication network $g \in \mathcal{G}^*$, for any random selection of two neighbors i and j there is a unique (stationary) outcome for the bargaining game where i initiates the negotiations and j is the first respondent (Proposition 1). At this unique equilibrium the neighbors i and j get the shares $(a_{ij}(g), 1 - a_{ij}(g))$ characterized by equations (2.1). In other words, when two neighbors i and j

¹⁰When the population is heterogeneous in time preferences, though, *ex post* payoffs do depend on players' locations. Selecting optimally the bargaining partner among neighbors becomes a relevant issue. See Calvó-Armengol (1999) for more details.

agree on a joint collaboration -player i being the promoter of the project- *ex post* payoffs are given by the equilibrium agreed-upon shares $(a_{ij}(g), 1 - a_{ij}(g))$.

According to our bargaining procedure, the initial proposer is chosen with uniform probability $1/n$. In words, an equalitarian flow of ideas about a new and profitable activity to be implemented jointly with an associate hits the population and all players have an equal opportunity to become a project promoter. Moreover, bargaining partners (potential associates) are drawn from neighborhoods with uniform probability. Given a network of communications $g \in \mathcal{G}^*$ we can thus easily compute, for any player $i \in N$, her expected payoff $Y_i(g)$ in a game where all players have an equal opportunity to become promoter of a business project and potential associates are treated equally. These individual payoffs define an allocation rule $Y : \mathcal{G}^* \rightarrow \mathbb{R}_+^n$ that gives the *ex ante* distribution of payoffs equal to the unique (stationary) expected equilibrium partition of the bargaining game with random selection of the negotiators. $Y(g)$ describes how individual (expected) payoffs are distributed given any bargaining network g connecting players. It is obviously related to the particular random device used to determine what player i is the promoter and to select her bargaining partner among the set $N(g, i)$ of potential associates. We remark, though, that by considering the expected subgame perfect equilibrium outcome with uniform probabilities the allocation rule we obtain is independent of which player initiates the bargaining procedure.

Proposition 2. *For all $g \in \mathcal{G}^*$, the allocation rule Y corresponding to the n -player bargaining game on g is given by:¹¹*

$$Y_i(g) = \frac{1}{n} \sum_{j \in N(g, i)} \left[\frac{1}{n(g, i)} a_{ij}(g) + \frac{1}{n(g, j)} (1 - a_{ji}(g)) \right], \forall i \in N.$$

Such an allocation rule is efficient: $\sum_{i \in N} Y_i(g) = 1$.

Given a communication structure $g \in \mathcal{G}^*$, for all player i in the population we can decompose her expected payoffs given by the allocation rule $Y(g)$ into two contributions: the *proposer (promoter) part* and the *respondent (associate) part*. The proposer part corresponds to the benefits player i can expect to obtain acting as a promoter of a business project and seeking for an associate among her set of

¹¹More generally, assume that for all $i \in N$ and $j \in N(g, i)$, i selects j as a co-bargainer with probability p_{ij} , where $0 \leq p_{ij} \leq 1$ and $\sum_{j \in N(g, i)} p_{ij} = 1$. Assume further that i is the promoter with probability $0 \leq q_i \leq 1$, where $\sum_{i \in N} q_i = 1$, and that these probabilities are independent. The allocation rule is then $Y_i(g) = \sum_{j \in N(g, i)} [q_i p_{ij} a_{ij}(g) + q_j p_{ji} (1 - a_{ji}(g))], \forall i \in N$.

acquaintances. The respondent part captures the expected flow of payoffs player i receives when collaborating as an associate to a business activity promoted by any of her neighbors.

$$Y_i(g) = \underbrace{\frac{1}{n} \sum_{j \in N(g,i)} \frac{1}{n(g,i)} a_{ij}(g)}_{\text{proposer (promoter) part}} + \underbrace{\frac{1}{n} \sum_{j \in N(g,i)} \frac{1}{n(g,j)} (1 - a_{ji}(g))}_{\text{respondent (associate) part}}$$

We know from Corollary 1 that the stationary subgame perfect equilibrium shares when players have a common discount factor $\delta \in (0, 1)$ coincides with the standard agreement $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$. Consequently, with homogeneous time preferences the allocation rule takes a particularly simple form. Denote by Y^h the allocation rule when the population is homogeneous in time preferences:

Corollary 2. *For all $g \in \mathcal{G}^*$, when the population is homogeneous in time preferences with common discount factor $\delta \in (0, 1)$, the allocation rule is given by:*

$$Y_i^h(g) = \frac{1}{1+\delta} \frac{1}{n} \left[1 + \delta \sum_{j \in N(g,i)} \frac{1}{n(g,j)} \right], \forall i \in N.$$

Even though *ex post* payoffs $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ are immune to players relative locations, *ex ante* payoffs as given by $Y^h(g)$ strongly depend on the frequency of activation of pairs of neighbors, hence on players' positions. They capture the asymmetries induced by the geometry of links in g .¹² We restrict from now on to the case of an homogeneous population with common discount factor $\delta \in (0, 1)$ for all players in N .

3.2. Definition of the bargaining power measure

As noted in the introduction, our model is aimed at determining how the place of a player in the network of communications affects her bargaining position with respect to the others. For the bargaining game considered, the discount factor δ measures the cost to delay (through disagreement) and thus introduces a friction in the model. By letting δ converge to unity this effect of discounting is removed from

¹²This allocation rule shares some features with the co-author model of Jackson and Wolinsky (1996) where the utility of some researcher i endowed with a fixed unit of time and involved in different collaborations with researchers in $N(g, i)$ -where a link in g represents bilateral collaboration between two researchers- is $u_i(g) = 1 + \left[1 + \frac{1}{n(g,i)}\right] \sum_{j \in N(g,i)} \frac{1}{n(g,j)}$.

the game: the time taken to formulate successive proposals becomes negligibly small, and the bargaining outcome is independent of the identity of the first proposer. The properties of the allocation rule when $\delta \rightarrow 1$ are then solely due to the communication network supporting the pairwise bargaining contests.

For all $g \in \mathcal{G}^*$, denote by $\Phi(g) \equiv \lim_{\delta \rightarrow 1} Y^h(g)$ the allocation rule in the limit $\delta \rightarrow 1$:

$$\Phi_i(g) = \frac{1}{2^n} \left[1 + \sum_{j \in N(g,i)} \frac{1}{n(g,j)} \right], \forall i \in N.$$

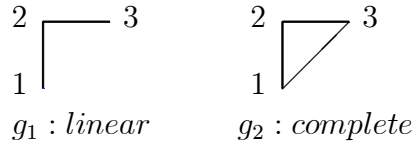
Proposition 3. *The allocation rule $\Phi(g)$ coincides with the asymmetric Nash bargaining solution where, for all $i \in N$, player i has bargaining power $\Phi_i(g)$, the set of agreements is $\mathcal{B}_n = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n : x_1 + \dots + x_n \leq 1\}$ and the disagreement point is 0_n .*

We can therefore state the following meaningful definition:

Definition 3. *For all $g \in \mathcal{G}^*$ and for all $i \in N$, $\Phi_i(g)$ is the bargaining power of player i in the communication network g .*

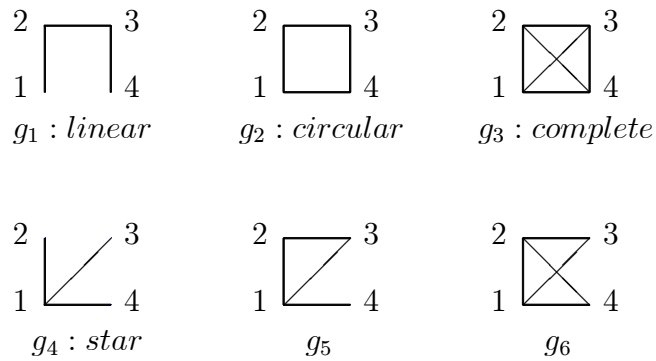
Given a graph $g \in \mathcal{G}^*$, the n -dimensional vector $\Phi(g)$ depends solely on the geometry of the communication network g , thus fulfilling the requirements for a bargaining power measure as stated by Binmore (1998): The bargaining powers in a weighted Nash bargaining solution should (...) not be interpreted in terms of bargaining skills. Bargaining powers are determined by the strategic advantages conferred on players by the circumstances under which they bargain. In our case, the network of communications. We now apply these results to some particular cases:

Example 1. $N = \{1, 2, 3\}$. *The possible connected communication networks are:*



	Player 1	Player 2	Player 3
g_1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
g_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Example 2. $N = \{1, 2, 3, 4\}$. The possible connected communication networks are:



	Player 1	Player 2	Player 3	Player 4
g_1	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{3}{16}$
g_2	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
g_3	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
g_4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
g_5	$\frac{18}{48}$	$\frac{11}{48}$	$\frac{11}{48}$	$\frac{8}{48}$
g_6	$\frac{7}{24}$	$\frac{7}{24}$	$\frac{5}{24}$	$\frac{5}{24}$

Example 3. Any finite population N with $n \geq 5$.

	Player 1	2	3	...	$n-2$	$n-1$	Player n
<i>linear</i>	$\frac{3}{4n}$	$\frac{5}{4n}$	$\frac{1}{n}$...	$\frac{1}{n}$	$\frac{5}{4n}$	$\frac{3}{4n}$
<i>circular</i>	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$...	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
<i>complete</i>	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$...	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
<i>star (centered on 1)</i>	$\frac{1}{2}$	$\frac{1}{2(n-1)}$	$\frac{1}{2(n-1)}$...	$\frac{1}{2(n-1)}$	$\frac{1}{2(n-1)}$	$\frac{1}{2(n-1)}$

4. Properties of the bargaining power measure

We establish some basic properties of the bargaining power measure $\Phi : \mathcal{G}^* \rightarrow \mathbb{R}_+^n$. To do so we first recall the definition of anonymity stated in Jackson and Wolinsky (1996). Given a permutation (isomorphism) $\pi : N \rightarrow N$ and a communication network $g \in \mathcal{G}^*$, let

$$g^\pi = \{\{i : j\} \mid i = \pi(k), j = \pi(l), \{k : l\} \in g\}.$$

We say that the bargaining power measure Φ is anonymous if, for any permutation π of N , $\Phi_{\pi(i)}(g^\pi) = \Phi_i(g)$. Anonymity states that, if we change the names of the players, their bargaining power according to Φ does not change. In other words, the label of the individuals does not matter and the only information required to determine $\Phi(g)$ is the particular shape of the communication network g . Let $g \in \mathcal{G}^*$.

Property 1 (Efficiency and Anonymity). The bargaining power $\Phi(g)$ is efficient and anonymous.¹³

Property 2 (Monotonicity). For all $i \in N$, the bargaining power $\Phi_i(g)$ of player i increases with the size of her neighborhood and decreases with the size of her neighbors' neighborhood.

¹³Efficiency is just a normalization condition requiring that $\sum_{i \in N} \Phi_i(g) = 1$. The efficiency of Φ is just a consequence of the efficiency of Y established in Proposition 2.

In words, the more player i has neighbors, the more she occupies a central position in the communication network and, consequently, the bigger is her bargaining power. Consider now any of i 's neighbors denoted by j . By increasing the size of j 's neighborhood, player j increases her bargaining power at the expenses of player i whose position in the network is weakened: i 's bargaining power decreases. Note that we do not need to assume that the total size of the population is fixed for the monotonicity property to hold: additional neighbors can be obtained either by establishing new communication channels within the population N of players, or by building links with new (outside) players thus increasing the size of the total population. Properties 1 and 2 also hold for the allocation rule Y^h corresponding to the case of a population homogeneous in time preferences where frictions are not removed.

Property 3 (Upper Bound). For all $i \in N$, $\Phi_i(g) \leq \frac{1}{2}$, and $\Phi_i(g) = \frac{1}{2}$ if and only if g is a star-shaped graph centered on i .¹⁴

Let $g \in \mathcal{G}^*$ and $i, j \in N$ such that $g \setminus \{i : j\} \in \mathcal{G}^*$.¹⁵ Denote by $D^{ij}\Phi(g) = \Phi(g) - \Phi(g \setminus \{i : j\})$ the *vector of marginal contributions* of the link $\{i : j\}$ to the players' bargaining power when the communication network is g .

Property 4 (Local Impact). $D_k^{ij}\Phi(g) = 0$, $\forall k \notin N(g, i) \cup N(g, j) \cup \{i, j\}$.

Therefore, adding or severing a link between two players only affects the bargaining power of these players and that of their direct neighbors. It has no effect out of the joint neighborhood of the two players. Recall that the bargaining power measure Φ is obtained as the limit when $\delta \rightarrow 1$ of the allocation rule Y when the population is homogeneous in time preferences. For all player i in the population, $Y_i(g)$ rule is naturally decomposed into a *proposer part* and a *respondent part*. We can also decompose player i 's bargaining power the same way:

$$\Phi_i(g) = \underbrace{\frac{1}{2n}}_{\text{proposer part}} + \underbrace{\frac{1}{2n} \sum_{k \in N(g, i)} \frac{1}{n(g, k)}}_{\text{respondent part}}$$

¹⁴ g is a star-shaped graph centered on i if and only if $N(g, i) = N \setminus \{i\}$ and $N(g, j) = \{i\}$, for all $j \neq i$.

¹⁵Which is equivalent to $n(g, i), n(g, j) \geq 2$. In particular, g cannot be a tree. The condition is trivially satisfied for all links $\{i : j\} \in g$ when g is a biconnected graph.

The *proposer part* is constant and independent of i 's location in the graph, while the *respondent part* depends both on i 's neighborhood $N(g, i)$ and on her neighbors' neighborhood $\{N(g, k)\}_{k \in N(g, i)}$. This latter contribution to a player's bargaining power is thus intimately related to the player position in the communication networks. Cutting the link $\{i : j\}$ modifies player i 's neighborhood $N(g, i)$. It thus affects player i 's bargaining power by altering her respondent part. But the neighborhood relationships are symmetric: $k \in N(g, i)$ is equivalent to $i \in N(g, k)$. Therefore, severing the link $\{i : j\}$ has also an effect on the respondent part of i 's neighbors thus affecting their bargaining power.

The property of Local Impact implies in particular that expanding or limiting the possibilities of communication only alters the bargaining power locally. The following two properties establish how the bargaining power measure is actually modified at the local level.

Property 5 (Weighted Fairness). $\frac{1}{n(g, i)} D_i^{ij} \Phi(g) = \frac{1}{n(g, j)} D_j^{ij} \Phi(g)$.

When two players establish a communication channel, they can directly communicate thus interact: their bargaining power is modified. On the contrary, if the link $\{i : j\}$ is severed, players i and j cannot communicate any more, and this inability to communicate also affects directly their bargaining power. Property 5 states that the marginal impact of building (or cutting) a link is the same for the players located at the end of the new (or severed) edge in *per neighbor* terms. Indeed, the weights are equal to the inverse of the players' neighborhood size. These weights are not exogenously given but depend on the communication network considered. In that sense this concept of *weighted fairness* differs from the standard concept in cooperative game theory introduced by Myerson (1980) and Hart and Mas-Colell (1989).

Weighted Fairness is concerned with the direct impact on two players' bargaining power of disconnecting them. But the bargaining power of any player depends on her neighbors' bargaining power. Therefore, cutting the link between any two players also affects (indirectly) the bargaining power of these two players' neighbors. The resulting (indirect) impact on the bargaining power of these two players' neighbors is characterized by Property 6 (Fair Reallocation). Let $\varepsilon_{ki} = 1$ if $k \in N(g, i)$ and 0 otherwise.

Property 6 (Fair Reallocation). $D_k^{ij} \Phi(g) = -\frac{\varepsilon_{ki}}{n(g, i)-1} D_j^{ij} \Phi(g) - \frac{\varepsilon_{kj}}{n(g, j)-1} D_i^{ij} \Phi(g)$,
 $\forall k \in N(g, i) \cup N(g, j)$.

The interpretation is as follows: suppose for simplicity that players i and j have no neighbors in common (disjoint neighborhoods). We restrict attention to the neighborhood of player i .¹⁶ The property of Fair Reallocation states that the indirect marginal contribution of the link $\{i : j\}$ to the bargaining power of the neighbors of player i is the same for all these neighbors. Indeed, for all $k \in N(g, i)$, $k \neq j$, $D_k^{ij}\Phi(g) = -\frac{1}{n(g,i)-1}D_j^{ij}\Phi(g)$. In words, the local reallocation of bargaining power resulting from adding or severing a link is fair: all neighbors are treated equally. In fact, as players only interact locally they do not discriminate among their neighbors according to their relative position in the communication network and, consequently, treat them on an equal footing. When players i and j have disjoint neighborhoods, Property 6 also implies that $\sum_{k \in N(g,i)} \Phi_k(g \setminus \{i : j\}) = \sum_{k \in N(g,i)} \Phi_k(g)$. Therefore, the total bargaining power of player i 's neighborhood remains unchanged when agent j is expelled from this neighborhood. Severing a link yields to a reallocation of the bargaining power within the neighborhood while preserving the total power of this neighborhood. In a sense, there is local conservation of power.

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¹⁶The case of agent j is deduced by symmetry.

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A. Appendix

Proof of Proposition 1. We first establish the equations characterizing the set of stationary subgame perfect equilibria outcomes, and then determine uniqueness. To do so we focus on the stationary strategies supporting these equilibria. A stationary strategy is simple as the actions it prescribes in every period do not depend on time nor on the events in the previous periods. For a given player $i \in N$, such a stationary strategy, denoted by σ_i , consists of:

- a set of proposals $(x_{ij}, 1 - x_{ij})$ made by i to her neighbors $j \in N(g, i)$;
- a set of responses (acceptance or refusal) to the offers made to i by her neighbors $j \in N(g, i)$: i accepts neighbor j 's offer of a share $1 - y$ if and only if $1 - y \geq 1 - y_{ji}$.

Consider an n -tuple stationary strategy $(\sigma_1, \dots, \sigma_n)$. We establish conditions on $(x_{ij})_{i \in N, j \in N(g, i)}$ and $(y_{ij})_{i \in N, j \in N(g, i)}$ such that $(\sigma_1, \dots, \sigma_n)$ is a stationary subgame perfect equilibrium. Suppose that for some $t = 0, 1, \dots$ player j rejects player i 's offer at round t (with $j \in N(g, i)$). Then at round $t + 1$ player j offers a share $1 - y_{jk}$ to the randomly selected neighbor $k \in N(g, j)$ that accepts it. Thus j gets an expected discounted payoff of $\delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} y_{jk}$. So, in order for player j 's rejection of every offer made by player i at round t of a share $1 - x < 1 - x_{ij}$ to be credible, we must have $1 - x_{ij} \leq \delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} y_{jk}$. At the same time we must have $1 - x_{ij} \geq \delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} y_{jk}$, otherwise player j would have an incentive to reject player i 's offer of a share $1 - x_{ij}$ at round t and to wait for the discounted expected payoff she gets in the continuation game. Therefore, $1 - x_{ij} = \delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} y_{jk}$, $\forall i \in N$ and $\forall j \in N(g, i)$.

We next prove that $x_{ij} = y_{ij}$, $\forall i \in N$ and $\forall j \in N(g, i)$. Indeed, player i knows that any of her neighbors $j \in N(g, i)$ accepts a share $1 - y_{ij}$, in which case player i gets a share of y_{ij} . Then necessary $x_{ij} \geq y_{ij}$. Also, player j accepting a share of $1 - x_{ij}$ requires that $1 - x_{ij} \geq 1 - y_{ij} \Leftrightarrow x_{ij} \leq y_{ij}$. Then necessary $x_{ij} = y_{ij}$. Thus a necessary condition for the n -tuple $(\sigma_1, \dots, \sigma_n)$ of stationary strategies to be a stationary subgame perfect equilibrium is that $1 - x_{ij} = \delta_j \frac{1}{n(g, j)} \sum_{k \in N(g, j)} y_{jk}$

and $x_{ij} = y_{ij}$, $\forall i \in N$, $\forall j \in N(g, i)$, implying equations (2.1). Reciprocally, any n -tuple $(\sigma_1, \dots, \sigma_n)$ satisfying equations (2.1) defines a stationary subgame perfect equilibrium where players offer what they can expect to get in the continuation game (appropriately discounted) and accept any proposal of at least this amount. Hence (2.1) fully characterize stationary subgame perfect equilibrium shares.

Establishing uniqueness of the stationary subgame perfect equilibrium outcome is now equivalent to prove that the system of equations $1 - a_{ij}(g) = \delta_j \frac{1}{n(g,j)} \sum_{k \in N(g,j)} a_{jk}(g)$, $\forall i \in N$ and $\forall j \in N(g, i)$ with unknowns $[a_{ij}(g)]_{i \in N, j \in N(g, i)}$ has a unique solution. Let $a_i(g) = \frac{1}{n(g,i)} \sum_{j \in N(g, i)} a_{ij}(g)$, $\forall i \in N$. Equations (2.1) can then be written as $1 - a_{ij}(g) = \delta_j a_j(g)$, $\forall i \in N$ and $\forall j \in N(g, i)$. Adding up within neighborhoods gives the equivalent system of equations:

$$\begin{cases} a_i(g) + \frac{1}{n(g,i)} \sum_{j \in N(g, i)} \delta_j a_j(g) = 1, \forall i \in N \\ a_{ij}(g) = 1 - \delta_j a_j(g), \forall i \in N \text{ and } \forall j \in N(g, i) \end{cases} \quad (\text{A.1})$$

The first set of equations of (A.1) can be written in matrix form $\mathcal{M} \cdot A = 1_n$, where $A = [a_1(g), \dots, a_n(g)]'$ and $1_n = [1, \dots, 1]'$ are vectors in \mathbb{R}^n , and the $n \times n$ matrix \mathcal{M} is given by $m_{ii} = 1 \forall i \in \{1, \dots, n\}$ and $m_{ij} = \frac{1}{n(g,i)} \delta_j$ if $j \in N(g, i)$ and 0 otherwise, $\forall j \neq i$. Then, $m_{ii} = 1 > \frac{1}{n(g,i)} \sum_{j \in N(g, i)} \delta_j = \sum_{j \neq i} m_{ij}$ meaning that the matrix \mathcal{M} has a dominant diagonal. Thus \mathcal{M} is invertible and the matrix system $\mathcal{M} \cdot A = 1_n$ has a unique solution $[a_i(g)]_{i \in N}$. Moreover the $[a_{ij}(g)]_{i \in N, j \in N(g, i)}$ are defined in an univocal way by the second set of equations (A.1). ■

Proof of Corollary 1. We easily check that the cake division $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ solves equations (2.1). ■

Proof of Proposition 2. Any player $i \in N$ only gets a positive share of the cake when she is either the initiator or the first respondent of the bargaining game. Then the payoff to a player i is the sum of:

- what she gets when she makes proposals to her neighbors: $\frac{1}{n} \frac{1}{n(g,i)} \sum_{j \in N(g, i)} a_{ij}(g)$;
- the flow of payoffs she receives from her neighbors when she accepts the proposals they make to her: $\frac{1}{n} \sum_{j \in N(g, i)} \frac{1}{n(g,j)} (1 - a_{ji}(g))$.

$$\text{Then, } \sum_{i \in N} Y_i(g) = \frac{1}{n} \left[\sum_{i \in N} \sum_{j \in N(g,i)} \frac{1}{n(g,i)} a_{ij}(g) + \sum_{i \in N} \sum_{j \in N(g,i)} \frac{1}{n(g,j)} (1 - a_{ji}(g)) \right]$$

$$\begin{aligned} &\Leftrightarrow \sum_{i \in N} Y_i(g) = \frac{1}{n} \left[\sum_{\{i:j\} \in g} \frac{1}{n(g,i)} a_{ij}(g) + \sum_{\{j:i\} \in g} \frac{1}{n(g,j)} (1 - a_{ji}(g)) \right] \\ &\Leftrightarrow \sum_{i \in N} Y_i(g) = \frac{1}{n} \sum_{\{i:j\} \in g} \frac{1}{n(g,i)} = \frac{1}{n} \sum_{i \in N} \sum_{j \in N(g,i)} \frac{1}{n(g,i)} = \frac{1}{n} \sum_{i \in N} 1 = 1. \end{aligned}$$

Therefore, $\sum_{i \in N} Y_i(g) = 1$, which concludes the proof. \blacksquare

Proof of Corollary 2. The expression for Y^h follows immediately from Corollary 1 and Proposition 2. \blacksquare

Proof of Proposition 3. We know from Proposition 2 that $\sum_{i \in N} Y_i^h(g) = 1$, $\forall \delta \in (0, 1)$. Moreover, $Y^h(\cdot)$ is continuous on $[0, 1]$. Hence at the limit $\delta \rightarrow 1$, $\sum_{i \in N} \Phi_i(g) = 1$.

Also, we can easily check that $[\Phi_1(g), \dots, \Phi_n(g)]' = \arg \max_{(x_1, \dots, x_n) \in \mathcal{B}_n} \prod_{i \in N} x_i^{\Phi_i(g)}$. \blacksquare

Proof of Property 1. Anonymity is clear. Efficiency has been established in the proof of Proposition 3 where we show that $\sum_{i \in N} \Phi_i(g) = 1$. \blacksquare

Proof of Property 2. We distinguish two different cases: total size of the population fixed, and increasing number of players in the population.

Case 1. Size of the population fixed: in that case, the size of a player's neighborhood increases by building a communication channel with a player in N with whom communication is not yet possible. Let $g \in \mathcal{G}^*$ and $i \in N$. Let $j \in N$ such that $\{i : j\} \notin g$.¹⁷ Let $g' = g \cup \{i : j\}$. Then $N(g', i) = N(g, i) \cup \{j\}$ and $N(g', j) = N(g, j) \cup \{i\}$, implying that $n(g', j) = n(g, j) + 1$ and $N(g', k) = N(g, k)$, $\forall k \neq i$ and $k \neq j$. We have,

$$\begin{aligned} \Phi_i(g') &= \frac{1}{2n} \left[1 + \sum_{k \in N(g', i)} \frac{1}{n(g, k)} \right] = \frac{1}{2n} \left[1 + \sum_{k \in N(g, i)} \frac{1}{n(g, k)} + \frac{1}{n(g, j) + 1} \right] \\ &\Leftrightarrow \Phi_i(g') = \Phi_i(g) + \frac{1}{2n} \frac{1}{n(g, j) + 1} \\ &\Rightarrow \Phi_i(g') > \Phi_i(g). \end{aligned}$$

¹⁷The statement is obvious for the complete graph g^N where $N(g^N, i) = N \setminus \{i\}$, $\forall i \in N$. We assume here that $g \neq g^N$.

Let $g^* \in \mathcal{G}^*$ such that $N(g^*, i) = N(g, i)$ and $N(g, j) \subset N(g^*, j)$, $\forall j \in N(g, i)$ with $n(g^*, l) > n(g, l)$ for at least one neighbor l of i . In words, the size of the neighborhood of at least one of i 's neighbors is strictly bigger in g^* than in g . Then,

$$\Phi_i(g^*) = \frac{1}{2n} \left[1 + \sum_{k \in N(g, i)} \frac{1}{n(g^*, k)} \right] > \frac{1}{2n} \left[1 + \sum_{k \in N(g, i)} \frac{1}{n(g, k)} \right] = \Phi_i(g). \quad Q.E.D.$$

Case 2. Size of the population increasing: in that case, the size of a player's neighborhood increases by building a communication channel with a player not yet in N . Denote such player by a and let $N' = N \cup \{a\}$: $|N'| = n + 1$. Let $g \in \mathcal{G}^*$ and $i \in N$. Let $g' = g \cup \{i : a\}$. Then $g' \in \mathcal{G}^*$, $N(g', i) = N(g, i) \cup \{a\}$, $N(g', a) = \{i\}$ and $N(g', j) = N(g, j)$, $\forall j \neq i$ and $j \neq a$, implying that $n(g', i) = n(g, i) + 1$, $N(g', a) = 1$ and $n(g', j) = n(g, j)$, $\forall j \neq i$ and $j \neq a$. Then,

$$\begin{aligned} \Phi_i(g') &= \frac{1}{2(n+1)} \left[1 + \sum_{j \in N(g', i)} \frac{1}{n(g', j)} \right] = \frac{1}{2(n+1)} \left[1 + \sum_{j \in N(g, i)} \frac{1}{n(g, j)} + 1 \right] \\ &\Leftrightarrow \Phi_i(g') - \Phi_i(g) = -\frac{1}{2n(n+1)} \left[1 + \sum_{j \in N(g, i)} \frac{1}{n(g, j)} \right] + \frac{1}{2(n+1)} \\ &\Leftrightarrow \Phi_i(g') - \Phi_i(g) = \frac{1}{n+1} \left[\frac{1}{2} - \Phi_i(g) \right] \end{aligned}$$

We know from Property 3 (proved thereafter) that for all $i \in N$, $\Phi_i(g) \leq \frac{1}{2}$ and $\Phi_i(g) = \frac{1}{2}$ if and only if g is a star-shaped graph centered on i . Therefore, $\Phi_i(g') \geq \Phi_i(g)$, $\forall i \in N$ and $\Phi_i(g') = \Phi_i(g)$ if and only if g' and g are star-shaped graphs centered on i . Let now $j \in N(g, i)$. We have $N(g', j) = N(g, j)$ and $N(g', k) = N(g, k)$, $\forall k \in N(g, j) \setminus \{i\}$, implying that $n(g', j) = n(g, j)$ and $n(g', k) = n(g, k)$, $\forall k \in N(g, j) \setminus \{i\}$. Then,

$$\begin{aligned} \Phi_j(g') &= \frac{1}{2(n+1)} \left[1 + \sum_{k \in N(g', j)} \frac{1}{n(g', k)} \right] = \frac{1}{2(n+1)} \left[1 + \sum_{k \in N(g, j)} \frac{1}{n(g, k)} + \frac{1}{n(g, i) + 1} \right] \\ &\Leftrightarrow \Phi_j(g') - \Phi_j(g) = -\frac{1}{n+1} \Phi_j(g) + \frac{1}{2} \left[\frac{1}{(n+1)(n(g, i) + 1)} - \frac{1}{n \cdot n(g, i)} \right] \end{aligned}$$

Therefore, $\Phi_j(g') < \Phi_j(g)$, $\forall j \in N(g, i)$. Q.E.D. ■

Proof of Property 3. For all $j \in N$, $1 \leq n(g, j) \Leftrightarrow \frac{1}{n(g, j)} \leq 1$ and $n(g, j) \leq n - 1$. Therefore, $\Phi_i(g) = \frac{1}{2n} \left[1 + \sum_{j \in N(g, i)} \frac{1}{n(g, j)} \right] \leq \frac{1}{2n} [1 + (n - 1)] = \frac{1}{2}$. Moreover,

$\Phi_i(g) = \frac{1}{2}$ for some i is equivalent to $\sum_{j \in N(g,i)} \frac{1}{n(g,j)} = n - 1$, possible if and only if $N(g,i) = N \setminus \{i\}$ and $N(g,j) = \{i\}$ for all $j \neq i$ that is, g is a star centered on i . ■

Proof of Property 4. Straightforward. ■

Proof of Property 5. For all $i, j \in N$ such that $g \setminus \{i : j\} \in \mathcal{G}^*$, $\Phi_i(g) - \Phi_i(g \setminus \{i : j\}) = D_i^{ij} \Phi(g) = \frac{1}{2n} \frac{1}{n(g,j)}$. Similarly we get $D_j^{ij} \Phi(g) = \frac{1}{2n} \frac{1}{n(g,i)}$. Therefore, $\frac{1}{n(g,i)} D_i^{ij} \Phi(g) = \frac{1}{n(g,j)} D_j^{ij} \Phi(g)$. ■

Proof of Property 6. Suppose for simplicity that i and j have no neighbors in common (disjoint neighborhoods). The proof for the general case is roughly the same but involves tedious notations. Let $k \in N(g,i)$, $k \neq j$. The bargaining power of player k is:

$$\Phi_k(g) = \frac{1}{2n} \left[1 + \sum_{l \in N(g,k)} \frac{1}{n(g,l)} \right] = \frac{1}{2n} \left[1 + \sum_{l \in N(g,k) \setminus \{i\}} \frac{1}{n(g,l)} + \frac{1}{n(g,i)} \right].$$

Consider the graph $g \setminus \{i : j\}$: $n(g \setminus \{i : j\}, i) = n(g, i) - 1$ and $n(g \setminus \{i : j\}, l) = n(g, l)$, $\forall l \in N(g, k) \setminus \{i\}$. Therefore, the bargaining power of player k with the communication structure $g \setminus \{i : j\}$ is:

$$\begin{aligned} \Phi_k(g \setminus \{i : j\}) &= \frac{1}{2n} \left[1 + \sum_{l \in N(g \setminus \{i : j\}, k) \setminus \{i\}} \frac{1}{n(g \setminus \{i : j\}, l)} + \frac{1}{n(g \setminus \{i : j\}, i)} \right] \\ &\Leftrightarrow \Phi_k(g \setminus \{i : j\}) = \frac{1}{2n} \left[1 + \sum_{l \in N(g, k) \setminus \{i\}} \frac{1}{n(g, l)} + \frac{1}{n(g, i) - 1} \right]. \end{aligned}$$

The differences of the two expressions for the bargaining power gives:

$$D_k^{ij} \Phi(g) = \frac{1}{2n} \left(\frac{1}{n(g, i)} - \frac{1}{n(g, i) - 1} \right) = -\frac{1}{n(g, i) - 1} \left(\frac{1}{2n} \frac{1}{n(g, i)} \right)$$

We know from the proof of Property 5 that $D_j^{ij} \Phi(g) = \frac{1}{2n} \frac{1}{n(g, i)}$, which concludes the proof. ■