Stable and Efficient Bargaining Networks*

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Abstract

We analyze the formation of undirected networks when individuals trade off the costs of forming and maintaining links against the bene ts from doing so. In our setting, the network connecting agents represents bargaining possibilities. Individual payoffs are computed as the unique expected outcome of a game where a player chosen randomly has an idea about a new and pro table economic activity to be implemented jointly with an associate with whom the pro t-sharing scheme is determined through noncooperative bargaining. We characterize stable and efficient networks, specify their geometry and determine when these two sets coincide.

Keywords: bargaining; network; stability; efficiency

JEL Classi cation: C72; C78; D20

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1 Introduction

This paper analyzes the formation of undirected networks when individuals trade off the costs of forming and maintaining links against the benefits from doing so. In our setting, the network connecting agents represents bargaining possibilities. The game we consider is the following. A player (the promoter) has an idea about a new and profitable economic activity to exploit jointly with an associate. The set of potential partners the promoter has at her disposal, presumably limited, is constituted by the players in the population she is in direct contact with. The promoter selects one player among her acquaintances, proposes this player to become her associate, and engages a bilateral negotiation to fix their respective shares of the outcome delivered by the new business activity. If the two players agree on how to share the benefits the negotiation ends, the firm is created, and its profits are distributed according to the agreed shares. Of course, these shares depend on the players preferences and on the negotiation rules. Moreover, as players may hold asymmetric positions in the network of bargaining possibilities, the issue of optimal bargaining partner selection has also to be contemplated.

We first solve the partner selection and bargaining game. In particular, we show that when the population is homogeneous in time preferences, two players that agree on a joint collaboration always share their benefits according to the standard half-half splitting rule. Players are identical *ex post* from the bargaining game viewpoint and the bargaining agreement they reach is immune to relative locations of players in the network. Individual payoffs are then computed as the unique expected outcome of the bargaining game on the network. They define a rule on the network that gives the *ex ante* distribution of payoffs as a function of players' positions, equal to the players' expected benefits less their costs of link formation. This allocation rule describes how individual (expected) payoffs are distributed given any bargaining network connecting players.

We then characterize stable and efficient networks, specify their geometry and determine when these two sets coincide. A network is pairwise stable if no player would benefit by severing an existing link, and no two players would benefit by forming a new link. In other words, stable networks are such that any player that is directly linked to another player has a strict incentive to maintain this link, and any two players that are not directly linked have no strict incentive to form a direct link with each other.[1] The concept of efficiency used throughout the paper is straightforward: a network is efficient if it maximizes the net aggregate bene t. We show that stable networks always exist and are fully determined in terms of neighborhood sizes. We then provide a direct characterization of the geometric architecture of symmetric stable networks that may be connected or disconnected, but are always payoff equivalent. Finally, we show that pairwise stable networks tend to be over-connected from a social viewpoint. This con ict between stability and efficiency is a common feature of the literature on social and economic networks both for directed (two-sided) graphs [e.g. Goyal (1993), Jackson and Wolinsky (1996)] and for undirected (one-sided) networks [e.g. Bala and Goyal (1999), Jackson (1999)].

The rest of the paper is organized as follows. Section 2 introduces the bargaining networks, describes the noncooperative bargaining game and solves it. Section 3 characterizes stable and efficient bargaining networks, addresses the question of their existence and discusses when these two sets coincide. Long proofs are presented in appendix.

2 Bargaining on networks

2.1 The network of bargaining possibilities

In many social and economic situations, agents are in direct contact with only a limited subset of other agents. When this is the case, bilateral volunteer meetings are limited to those pair of agents that can communicate with each other directly (or that know each other directly). We represent these restrictions on the feasible pairwise meetings by a graph where the nodes are identi ed with the players and a link between two nodes means that the corresponding players can bargain bilaterally. The network connecting players thus represents the bargaining possibilities.

[1] See Dutta and Mutuswami (1997) and Jackson and Wolinsky (1996) for a formal de nition and thorough discussion of this stability concept.

Let $N = \{1, \ldots, n\}$ be the finite set of players ($n \geq 2$). We only consider undirected graphs. The complete graph, denoted g^N, is thus the set of all subsets of N of size 2, and the set of all possible graphs on N is $\mathcal{G} \equiv \{g \mid g \subseteq g^N\}$. Let $g \in \mathcal{G}$. A link in g between two players i and j is denoted by ij, where $ij \in g$. We introduce the following definition:

Definition 1 *For all $g \in \mathcal{G}$ and for all $i \in N$, i s neighborhood denoted by $N(g, i)$ is the set of players in N directly connected to i when the bargaining network is g that is, $N(g, i) = \{j \in N \setminus \{i\} \mid ij \in g\}$.*

A similar type of neighborhood structure can be found for instance in Bala and Goyal (1998). Note that by definition $i \notin N(g, i)$, and $i \in N(g, j)$ is equivalent to $j \in N(g, i)$. In words, the neighborhood relationship is non reflexive and symmetric. We denote by $n(g, i)$ the cardinality of $N(g, i)$.[2] To avoid trivialities we will restrict to $\mathcal{G}^* = \{g \in \mathcal{G} \mid n(g, i) \geq 1, \forall i \in N\}$ that is, to those networks where all players have at least one neighbor.

2.2 The bargaining game

An informal version of the game we consider is the following. Suppose that some player (the promoter) has an idea about a new and profitable economic activity to exploit jointly with an associate. The set of potential partners the promoter has at her disposal, presumably limited, is constituted by the players in the population she is in direct contact with. The promoter selects one player among her acquaintances, proposes this player to become her associate, and engages a bilateral negotiation to fix their respective shares of the outcome delivered by the new business activity. If the two players agree on how to share the benefits the negotiation ends, the firm is created, and its profits are distributed according to the agreed shares. Of course, these shares depend on the players preferences and on the negotiation rules. Moreover, as players may hold asymmetric positions in the network of bargaining possibilities, the issue of optimal bargaining partner selection has also to be contemplated.

[2] In graph theory, $n(g, i)$ is the degree of node i that is, the number of nodes adjacent to i or, equivalently, the number of links that are incident with i.

We model this bargaining game as a two-stage game. In the first stage, players choose independently to whom they wish to submit bargaining proposals for a potential collaboration in a joint project. In the second stage, one player randomly chosen has an idea about a business opportunity to exploit jointly with an associate and initiates a round of negotiations with her selected partner.[3] In case of rejection, the potential associate submits a proposal to her selected partner that may coincide with the initial promoter or not. We first introduce a definition:

Definition 2 *For all $g \in \mathcal{G}^*$, $\tau_g : N \longrightarrow N$ where $\tau_g(i) \subseteq N(g,i)$, $\forall i \in N$ is the selection correspondence on g that gives for any player i the subset $\tau_g(i) \subseteq N(g,i)$ of neighbors player i chooses to bargain with.*

Given a connected undirected bilateral graph g on N, the two-stage game we analyze is the following:

> *Stage One.* For all $i \in N$, player i determines the subset $\tau_g(i)$ of neighbors in $N(g,i)$ she chooses to bargain with.
>
> *Stage Two.* Noncooperative bilateral negotiation with random selection of the promoter $i \in N$, two-player/one-cake bargaining offer at the first round by the initial proposer i to her respondents in $\tau_g(i)$. In case of refusal, sequential activation of pairs of bargaining partners from neighborhood to neighborhood according to τ_g.

We assume from now on that for all $i \in N$, $\tau_g(i)$ is a singleton.[4] Any selection function τ_g defines a sequence of proposers and respondents for the bargaining game on the graph g the following way. Denote by p_t the label of the proposer at some period $t \in \mathbb{N}$. Suppose that player p_t makes an offer to

[3] Other examples of two-stage network formation games can be found in Aumann and Myerson (1988), Kranton and Minehart (1998) and Qin (1996). A related paper is Slikker and van den Nouweland (1999).

[4] We can assume for instance that the following tie-breaking rule applies: when some player $i \in N$ is indifferent between neighbors $j_1, \ldots j_m \in N(g,i)$ (concerning the outcome she can obtain bargaining with any of them separately), the respondent with the smallest index is selected, that is $\tau_g(i) = \min\{j_1, \ldots j_m\}$.

her bargaining partner (neighbor) $\tau_g(p_t)$. If the offer is accepted the game ends. Otherwise the rejector $\tau_g(p_t)$ becomes the new proposer at the next round $t+1$ that is, $p_{t+1} = \tau_g(p_t)$. And so on. The game proceeds until, and if, a bilateral agreement between two neighboring players is reached. Therefore, the noncooperative bargaining game on the communication structure actually played is fully described by the identity of the initiator $i = p_0$ of the game and the selection function τ_g. We denote by $\langle \tau_g, i \rangle$ such a game.

2.3 Analysis of the game

Assume that the population is homogeneous in time preferences and denote by $\delta \in (0,1)$ the common discount factor. Let $g \in \mathcal{G}^*$.

Proposition 1 *For all promoter $i \in N$ and for all selection function $\tau_g : N \to N$, the game $\langle \tau_g, i \rangle$ has a unique perfect equilibrium outcome corresponding to the standard agreement $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$. In particular, the optimally selected partner $\tau_g(i)$ can be indistinctly any player in $N(g,i)$.*

Proof. See the appendix. □

Let δ converge to unity. Proposition 1 says that when two neighbors agree on a joint collaboration in a project, the derived bene ts are distributed according to a standard half-half splitting rule irrespective of their eventual asymmetric locations in the graph. In other words, whoever the promoter, all her neighbors (either isolated or well-connected) are worth the same *ex post* as potential associates.

We can therefore assume that every player $i \in N$ treats all her neighbors on an equal footing and selects her bargaining partner within her neighborhood with uniform probability. Formally, let

$$\Pr\{\tau_g(i) = j\} = \tfrac{1}{n(g,i)}, \forall j \in N(g,i).$$

Assume further that all players in the population have the same probability $\frac{1}{n}$ to become a promoter (land of equal opportunity) and denote by $V > 0$ the rm pro t. An important feature of our model is that links are costly. Each direct link ij results in a cost $c > 0$ to both i and j, equal across individuals players and independent of the number of existing links. This cost of forming a link with another player may for instance be interpreted as the time, the effort or the money a player must spend in order to maintain an active connection.[5] Given a bargaining network $g \in \mathcal{G}^*$ we can then easily compute for any player i her expected net payoff $Y_i(g)$ from the game above

$$Y_i(g) = \frac{V}{2n}\left[1 + \sum_{j \in N(g,i)} \frac{1}{n(g,j)}\right] - cn(g,i),\ \forall i \in N.$$

These individual payoffs de ne an allocation rule $Y : \mathcal{G}^* \to \mathbb{R}^n_+$ that gives the *ex ante* distribution of payoffs, equal to the players expected bene ts less their costs of link formation. $Y(g)$ describes how individual (expected) payoffs are distributed given any bargaining network g connecting players. It is obviously related to the particular random device used to determine what player i is the promoter, and to select her bargaining partner $\tau_g(i)$ among the set $N(g,i)$ of potential associates.[6] From this expression we can deduce that even though *ex post* payoffs are immune to players relative locations, *ex ante* payoffs as de ned here strongly depend on the frequency of activation of pairs of neighbors, hence on players positions. They capture the asymmetries induced by the geometry of links.[7]

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[5] We could normalize $V = 1$ or, equivalently, $c = 1$ without loss of generality. Indeed, it is the rate of the rm value V to the per-capita cost of a link c that matters when characterizing both stable and efficient networks. We will, though, keep both (redundant) parameters V and c throughout to clearly differentiate gains from costs when considering the individual incentives to form and cut links.

[6] More generally, assume that for all $i \in N$ and for all $j \in N(g,i)$, $\Pr\{\tau_g(i) = j\} = p_{ij}$, where $0 \leq p_{ij} \leq 1$ and $\sum_{j \in N(g,i)} p_{ij} = 1$. Assume further that i is the promoter with probability $0 \leq q_i \leq 1$, where $\sum_{i \in N} q_i = 1$, and that these probabilities are independent. The expected payoffs are then $Y_i(g) = \frac{V}{2}\left[q_i + \sum_{j \in N(g,i)} q_j p_{ji}\right] - cn(g,i)$, $\forall i \in N$. The equal-opportunity and equal-treatment-of-neighbors random device analyzed in the paper does not introduce any additional bias in the payoffs and preserves those that are merely due to the asymmetries in locations. See Calvó-Armengol (1999b) for more details.

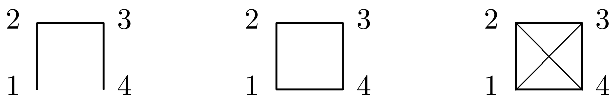
[7] This allocation rule shares some features with the co-author model of Jackson and Wolinsky (1996) where the utility of some researcher i endowed with a xed unit of time and involved in different collaborations with researchers in $N(g,i)$ (a link in g representing bilateral collaboration between two researchers) is given by $u_i(g) = 1 + \left[1 + \frac{1}{n(g,i)}\right] \sum_{j \in N(g,i)} \frac{1}{n(g,j)}$.

Example 1 $N = \{1, 2, 3\}$. *The possible connected bargaining networks and the resulting payoffs are:*

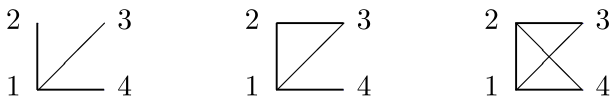
g_1 : *linear* g_2 : *complete*

	Player 1	Player 2	Player 3
g_1	$\frac{1}{4}V - c$	$\frac{1}{2}V - 2c$	$\frac{1}{4}V - c$
g_2	$\frac{1}{3}V - 2c$	$\frac{1}{3}V - 2c$	$\frac{1}{3}V - 2c$

Example 2 $N = \{1, 2, 3, 4\}$. *The possible connected bargaining networks and the resulting payoffs are:*



g_1 : *linear* g_2 : *circular* g_3 : *complete*



g_4 : *star* g_5 g_6

	Player 1	Player 2	Player 3	Player 4
g_1	$\frac{3}{16}V - c$	$\frac{5}{16}V - 2c$	$\frac{5}{16}V - 2c$	$\frac{3}{16}V - c$
g_2	$\frac{1}{4}V - 2c$	$\frac{1}{4}V - 2c$	$\frac{1}{4}V - 2c$	$\frac{1}{4}V - 2c$
g_3	$\frac{1}{4}V - 3c$	$\frac{1}{4}V - 3c$	$\frac{1}{4}V - 3c$	$\frac{1}{4}V - 3c$
g_4	$\frac{1}{2}V - 3c$	$\frac{1}{6}V - c$	$\frac{1}{6}V - c$	$\frac{1}{6}V - c$
g_5	$\frac{18}{48}V - 3c$	$\frac{11}{48}V - 2c$	$\frac{11}{48}V - 2c$	$\frac{8}{48}V - c$
g_6	$\frac{7}{24}V - 3c$	$\frac{7}{24}V - 3c$	$\frac{5}{24}V - 2c$	$\frac{5}{24}V - 2c$

3 Stability and efficiency of bargaining networks

3.1 Definitions

We use the concept of pairwise stability introduced by Jackson and Wolinsky (1996). It states that a network is pairwise stable if no player would benefit by severing an existing link, and no two players would bene t by forming a new link.

De nition 3 *The network $g \in \mathcal{G}^*$ is pairwise stable if and only if both:*

(a) for all $ij \in g$, $Y_i(g) \geq Y_i(g - ij)$ and $Y_j(g) \geq Y_j(g - ij)$;

(b) for all $ij \notin g$, $Y_i(g + ij) > Y_i(g)$ implies $Y_j(g + ij) < Y_j(g)$.

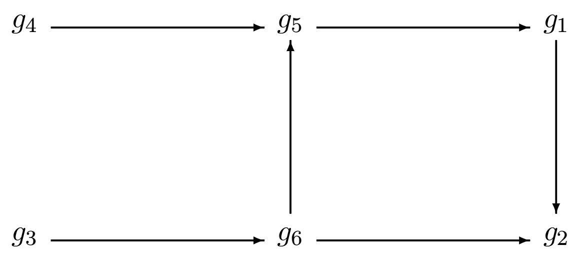
Stable networks, thus, are such that no player gains by altering the current con guration of links, neither by adding a new connection nor by eliminating an existing one. This notion of stability allows link severance only by individuals and link formation only by pairs of players. It is just a concept among many other possible notions that provides sharp results for our analysis.[8] In particular, the class of pairwise stable networks can be interpreted as the limiting graphs of a dynamic procedure of endogenous network formation.[9] Consider indeed a dynamic process were players myopically add or sever links based on the improvement the resulting network offers relative to the current network, with each network differing from the previous only by one link. Clearly, if this formation process converges, the networks ultimately reached are pairwise stable.

Example 3 $N = \{1, 2, 3, 4\}$. *Let* $V = 20c$. *The payoffs are:*

[8] One variation of this stability notion would be to strenghten the concept of pairwise stability by allowing for side payments between two players who deviate by constructing a new link. Another variation allows for a network to be immune to deviations by more than two players simultaneously (in the case of link formation), and to the severance of more than one link by a single player. Also, stable networks can be de ned as the Nash equilibria (and its re nements when required) of a one-shot strategic link formation game where players strategies consist on the set of players with whom they want to form a link. We refer the reader to Dutta and Mutuswami (1997) for a thorough analysis of the notion of (coalitional) stability as a Nash equilibria of a strategic form game when players are graph-connected.

[9] Jackson and Watts (1998) for the case of undirected graphs, and Bala and Goyal (1999) for the case of directed graphs directly tackle the question of network formation in a dynamic framework.

	Player 1	*Player 2*	*Player 3*	*Player 4*
g_1	$\frac{33}{12}c$	$\frac{51}{12}c$	$\frac{51}{12}c$	$\frac{33}{12}c$
g_2	$\frac{36}{12}c$	$\frac{36}{12}c$	$\frac{36}{12}c$	$\frac{36}{12}c$
g_3	$\frac{24}{12}c$	$\frac{24}{12}c$	$\frac{24}{12}c$	$\frac{24}{12}c$
g_4	$\frac{84}{12}c$	$\frac{28}{12}c$	$\frac{28}{12}c$	$\frac{28}{12}c$
g_5	$\frac{54}{12}c$	$\frac{31}{12}c$	$\frac{31}{12}c$	$\frac{28}{12}c$
g_6	$\frac{34}{12}c$	$\frac{34}{12}c$	$\frac{26}{12}c$	$\frac{26}{12}c$



A sequence of networks differing by just one link, where either the pair of players that agreed on the creation of a new link or the player that unilaterally severed an existing link are better off, is called a sequence of improving paths. In particular, $g \rightarrow g'$ means that g' improves g in the sense described above. This concept of improving path is introduced and extensively analyzed in Jackson and Watts (1998). In the example considered ($n = 4$ players and $V = 20c$), this sequence always converges to the circular graph g_2 which is the unique pairwise stable graph in this case (see Theorem 1 and Corollary 1).

The concept of efficiency used throughout the paper is due to Goyal (1993) and to Jackson and Wolinsky (1996). It says that a network is efficient if it maximizes the net aggregate bene t:

De nition 4 *A network $\overline{g} \in \mathcal{G}^*$ is efficient if and only if it maximizes the summation of each player payoff that is, $\bar{g} \in \arg\max_{g \in \mathcal{G}^*} \sum_{i \in N} Y_i(g)$.*

3.2 Stable networks: characterization

In this section we address the question of what the application of pairwise stability predicts concerning which graphs might form. We rst need a technical result:

Lemma 1 *$Y_i(g+ij) > Y_i(g)$ if and only if $n(g,j) < \frac{V}{2nc} - 1$.*

Proof. By de nition, $Y_i(g) = \frac{V}{2n}\left[1 + \sum_{k \in N(g,i)} \frac{1}{n(g,k)}\right] - cn(g,i)$. Then clearly, $Y_i(g+ij) = Y_i(g) + \frac{V}{2n}\frac{1}{n(g+ij,j)} - c$, where $n(g+ij,j) = n(g,j)+1$, which yields the desired result. □

In words, forming a new link is strictly bene cial for the two parties involved if and only if both parties are not too well-connected. The intuition for this result is the following. Player i s neighbors contribute to i s payoff by an amount inversely proportional to the size of their neighborhood: if k is in the neighborhood of player i, $k \in N(g,i)$, the contribution of player k to $Y_i(g)$ is given by $\frac{V}{2n}\frac{1}{n(g,k)}$. The more k is densely connected in g, the higher $n(g,k)$ and, consequently, the lower her marginal contribution to i s payoff. But the cost c incurred by player i to maintain the link with k is constant and independent of the particular geometry of links. Therefore, when player k s neighborhood size $n(g,k)$ is too high, this cost may not compensate player i for the bene t of being connected with k.

When players trade off the costs of forming and maintaining links against the potential bene ts from doing so, the parameters V and c clearly play a major role. In particular,

Proposition 2 *The complete graph g^N is the unique pairwise stable network if and only if $V \geq 2n(n-1)c$.*

Proof. The condition $V \geq 2n(n-1)c$ is equivalent to $n-2 \leq \frac{V}{2nc} - 1$. We then deduce from Lemma 1 that for any two players i and j such that $ij \notin g$ and $n(g,i)$, $n(g,j) \leq n-2$, it is mutually bene cial to establish a direct link between them. Reciprocally, g^N being pairwise stable implies again from Lemma 1 that $n-2 \leq \frac{V}{2nc} - 1$. □

The complete graph g^N has $n(n-1)$ different links. The total cost paid by players in N to be fully (intra)connected is thus equal to $2n(n-1)c$. Proposition 2 establishes that a sufficient condition for the complete graph to be pairwise stable is that the potential joint bene t V be strictly higher than the cost of maintaining full communication among the population. Moreover, when $V \geq 2n(n-1)c$ links are very cheap and players have an incentive to add every link and never delete a link, implying that g^N is the unique stable network. For low value of the per-link cost c or, equivalently, for high values of the rm pro t V, the unique stable network is fully connected and, consequently, the society is highly cohesive.

On the contrary, when $V < 2n(n-1)c$ the complete graph is not stable anymore. We can nonetheless identify stable con gurations of links that depend on the relative values of V and c, and on the population size n. Let $\eta(V,c) \equiv \left\lfloor \frac{V}{2nc} \right\rfloor$ denote the highest integer inferior or equal to $\frac{V}{2nc}$. For the rest of the paper we assume that $V < 2n(n-1)c$, implying in particular that $\eta(V,c) < n-1$.

Theorem 1 *Suppose that $V \geq 4nc$. A network $g \in \mathcal{G}^*$ is pairwise stable if and only if both:*

(a) for all $ij \in g$, $n(g,i) \leq \eta(V,c)$ and $n(g,j) \leq \eta(V,c)$;

(b) for all $ij \notin g$, $n(g,i) < \eta(V,c) - 1$ implies $n(g,j) \geq \eta(V,c) - 1$.

Proof. Follows from Lemma 1 and the de nition of pairwise stability. The condition $V \geq 4nc$ is equivalent to $\eta(V,c) \geq 2$, which in turn guarantees that $\eta(V,c) - 1 \geq 1$. □

Stable networks are thus fully determined in terms of neighborhood sizes. In that sense, Theorem 1 provides quite a simple characterization of stable graphs. In practice, though, the criteria for pairwise stability stated above yields to tedious and systematic link-by-link examination. We show in the next section that a direct characterization of some appealing networks is nonetheless possible. So far, we have assumed that $V \geq 4nc$, which is equivalent to requiring that all players have at least two neighbors in a stable con guration of links that is, $\eta(V,c) \geq 2$. In fact, we can relax this assumption and allow for players to have only one neighbor. This is the purpose of the following propositions.

Proposition 3 *Suppose that* $nc \leq V < 2nc$*. If* n *is even, the networks consisting of* $n/2$ *separate pairs is stable. If* n *is odd, the network consisting of* $(n-1)/2$ *separate pairs and one isolated player is stable.*

Proof. Within a separate pair, the individual payoffs are $\frac{V}{n} - c$. Adding a link yields to a payoff of $\frac{3V}{2n} - 2c$ for the central player with two different neighbors (resp. $\frac{3V}{4n} - c$ for any of her peripheral neighbors), hence results in a payoff reduction equal to $\frac{V}{2n} - c < 0$ (resp. $-\frac{V}{4n} < 0$). Also, severing an existing link yields to a null payoff thus reducing the payoff by $\frac{V}{n} - c \geq 0$.□

Remark 1 *We are assuming throughout that the bargaining network* g *belongs to* $\mathcal{G}^*$*, meaning that all players have at least one neighbor according to* g*. According to the previous proposition, when both* $nc \leq V < 2nc$ *and* n *is odd, the unique stable network consists of a bunch of separate pairs and a fully isolated player. Therefore, in this case, there exists no stable network in* $\mathcal{G}^*$*.*

Proposition 4 *Suppose that* $2nc \leq V < 4nc$*. The star-shaped networks consisting of one central player and* $n-1$ *peripheral players are stable.*

Proof. Let g^1 be the star-shaped network centered on player 1, meaning that $N(g^1, 1) = N \backslash \{1\}$ and $N(g, j) = \{1\}$, $\forall j \neq 1$. Then, $Y_1(g^1) = \frac{V}{2} - (n-1)c$ and $Y_j(g^1) = \frac{V}{2(n-1)} - c$, $\forall j \neq 1$. If two peripheral players $j \neq 1 \neq k$ establish a link between them they get $Y_j'(g^1) = Y_j(g^1) + \frac{V}{4n} - c < Y_j(g^1)$ (idem for player k). If some player $j \neq 1$ cuts the unique link she has with player 1 she ends up with a null payoff thus incurring a loss -$\frac{V}{2(n-1)} + c < 0$. Suppose now that the central player cuts an existing link. We are left with a star-shaped network with only $n-2$ peripheral players. The resulting payoff is $Y_1'(g^1) = Y_1(g^1) - \frac{V}{2n} + c \leq Y_1(g^1)$. □

3.3 The case of symmetric networks

A direct characterization of the architecture of some appealing networks is possible. Bala and Goyal (1999) introduce this notion of architecture that relates to the geometry of graphs, independently of the particular assignment of players to nodes (i.e. labeling of nodes). We say that two networks g, $g' \in$

$\mathcal{G}^*$ are equivalent if they have the same geometry, meaning that one graph can be deduced from the other just by changing the names of the players.[10] This equivalence relation partitions the set of possible graphs on N into classes, and each class is referred to as an architecture. Two networks with common architecture are geometrically identical. In particular, symmetric architectures are such that all neighbors have the same neighborhood size: for all $i, j \in N$, $n(g,i) = n(g,j)$.

The allocation rule $Y(g)$ de ned above is anonymous in that changing the names of the individuals while keeping the geometry of the network connecting them does not alter the payoffs they receive. In other words, the only information required to determine $Y(g)$ is the particular shape of g, and not the label of players. Moreover, all players are identical in the game considered. It is thus natural to concentrate on symmetric architectures. For the rest of the section we assume that $4nc \leq V < 2n(n-1)c$.

Corollary 1 *Symmetric architectures with neighborhood size $\eta(V,c)$ are pairwise stable. For all $2 \leq k \leq n-2$, when $2knc \leq V < 2(k+1)nc$ the symmetric architecture with neighborhood size k is stable.*

Proof. Symmetric architectures with neighborhood size $\eta(V,c)$ clearly satisfy the conditions for stability stated in Theorem 1. □

In the previous section we have been able to provide a general characterization of stable networks only in terms of neighborhood sizes. Indeed, Theorem 1 is a degree-type of result, where a certain property of networks (here, pairwise stability) is expressed in terms of the number of links adjacent to any player. Corollary 1 is just a restatement of this general theorem for the particular case of symmetric networks that is, graphs where every player has the same number of connections and relative locations are interchangeable. This corollary complements the results established in Propositions 2, 3 and 4 that altogether provide a full characterization of stable networks for

[10] We are thus requiring that there exists a one-to-one mapping from the nodes of g to the nodes of g' that preserves the adjacency of nodes. Formally, g and g' are equivalent if and only if there exists a permutation $\pi : N \to N$ such that $g' = g^\pi$ where $g^\pi \equiv \{ij \mid i = \pi(k), j = \pi(l), kl \in g\}$. In graph theory, such two networks are called isomorphic.

the full range of parameters where $V \geq nc$ that is, where it is individually bene cial for any player to set up the rst link. Summing up these results we obtain the following classi cation of stable networks. When,

- $nc \leq V < 2nc$, separate pairs are stable;
- $2nc \leq V < 4nc$, stars are stable;
- $2nkc \leq V < 2n(k+1)c$ for $2 \leq k < n-2$, $k-$symmetric networks are stable;
- $2n(n-1)c \leq V$, the complete graph is stable.

Corollary 2 *Stable symmetric architectures with neighborhood size $\eta(V,c)$ are payoff-equivalent. Individual payoffs are equal to $\frac{V}{2n}+c\rho$, where $\rho \in [0,1)$.*

Proof. Let g be a symmetric architecture. For all $i \in N$, the payoff of player i is $Y_i(g) = \frac{V}{2n}\left[1+\sum_{k\in N(g,i)}\frac{1}{\eta(V,c)}\right] - c\eta(V,c) = \frac{V}{n} - c\eta(V,c)$, where $\eta(V,c) = \left\lfloor \frac{V}{2nc} \right\rfloor$. Let $\rho = \frac{V}{2nc} - \left\lfloor \frac{V}{2nc} \right\rfloor$; by de nition $\rho \in [0,1)$. With some computations we obtain $Y_i(g) = \frac{V}{2n} + c\rho$. □

Therefore, when symmetric architectures of size $\eta(V,c)$ exist, they all reward the players the same amount $y = \frac{V}{2n} + c\rho$, implying that the per-capita payoff y is bounded above and below: $\frac{V}{2n} \leq y < \frac{V}{2n} + c$. Given a value $V > 0$ for the rm bene t and a constant cost $c > 0$ of link formation, players can explicitly compare the costs of forming and maintaining links against the bene ts from doing so. This trade-off at the individual level generates a sequence of improving networks that eventually leads to a stable con guration of links. Stable networks, thus, are such that no player gains by altering the current con guration of links, neither by adding a new connection nor by eliminating an existing one. Corollary 2 states that, given V and c, non-coordinated and pro t-seeking individual players organize themselves in a way that guarantees a per-capita payoff at least equal to $\frac{V}{2n}$. Individual incentives yield to a geometry of links that depends on the per-link cost through the neighborhood size $\eta(V,c)$, but that rewards a minimal payoff $\frac{V}{2n}$ independent of this cost. Players somehow manage to circumvent the coordination problem they face by generating a bargaining network that allows them to enjoy a payoff that increases linearly with the rm s bene t V.

Even though symmetric stable architectures are payoff equivalent, they may differ in their geometry. In particular, given V and c, stable networks may either be connected or disconnected. Connected networks are such that there is a path between every pair of nodes in the graph.[11] Disconnected networks are partitioned into two or more subsets (subgraphs) in which there are no paths between the nodes in different subsets. When these subgraphs are connected, they are called components. A component of a graph is thus a maximal connected subgraph.

Proposition 5 *A stable symmetric architecture with neighborhood size* $\eta(V,c)$ *has at most* $\left\lfloor \frac{n}{\eta(V,c)+1} \right\rfloor$ *disconnected subsets. In particular, if* $\frac{n}{\eta(V,c)+1} = S$ *for some* $S \in \mathbb{N}$*, the more disaggregated stable architecture consists of* S *components (connected subgraphs) of* $\eta(V,c)+1$ *players each.*[12]

Proof. The results follows from the observation that a symmetric architecture with size $\eta(V,c)$ has at least $\eta(V,c)+1$ players. □

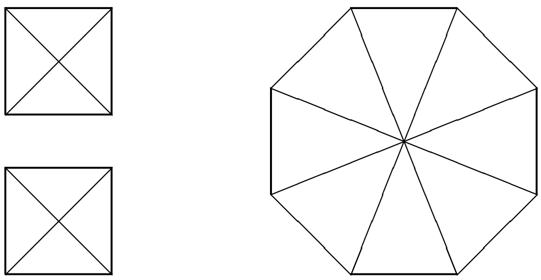
Stable symmetric architectures may either be connected or disconnected. For the case of connected networks, the higher the neighborhood size $\eta(V,c)$, the tighter the population s internal cohesiveness. Alternatively, the lower $\eta(V,c)$, the looser the population s internal cohesiveness. Moreover, the neighborhood size $\eta(V,c)$ depends positively on the potential bene t V and negatively on the cost of forming links c. Therefore, for low values of V (equivalently, high values of c), connected stable graphs are loose, implying that the society is cohesionless. As V increases (equivalently, as c decreases) stable networks are increasingly tight and the population internal cohesiveness is strengthened. The size of the society also matters: as n increases, cohesiveness decreases.

[11] A network g is connected if two players i and j in N either communicate directly ($ij \in g$) or are connected through a path of neighbors, that is there exists $p \geq 1$ and a sequence $\{i_0, \ldots, i_p\} \subset N$ such that $i_0 = i$, $i_p = j$, $i_{k-1} \neq i_k$ and $i_{k-1}i_k \in g$, $\forall 1 \leq k \leq p$.

[12] Note that the condition $n = C(\eta(V,c)+1)$ for some $C \in \mathbb{N}$ is equivalent to $\left\lfloor \frac{n}{\eta(V,c)+1} \right\rfloor = \frac{n}{\eta(V,c)+1}$.

Regarding disconnected networks, the higher the number of subcomponents a graph can be disaggregated on, the more fractionated the population. Alternatively, the lower the number of components, the more integrated the population. Proposition 5 gives an upper bound on the number of components (connected subgraphs) a stable network can be subdivided on inversely related to $\eta(V, c)$. Therefore, for low values of V (equivalently, high values of c), disconnected stable graphs can be very numerous, implying a high population fragmentation. As V increases (equivalently, as c increases) the fragmentation is alleviated and the society becomes more integrated. Again, the population size also matters: holding V and c xed, increasing n enhances the possibilities for fragmentation to occur.

Example 4 *Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *and* $V \in [48c, 64c)$. *Then* $\eta(V, c) = \left\lfloor \frac{V}{2nc} \right\rfloor = 3$. *The only two possible symmetric stable architectures of size* 3 *are:*



3.4 Stable networks: existence

In the previous sections we have fully characterized stable networks in terms of neighborhood sizes, with a special emphasis on symmetric stable networks. But so far, nothing has been said regarding their existence. This issue may be irrelevant in some cases. In particular the stable networks described by Proposition 2 (the complete graph when $V \geq 2n(n-1)c$), Proposition 3 (separate pairs when $nc \leq V < 2nc$) and Proposition 4 (stars when $2nc \leq V < 4nc$) trivially exist. Yet, the existence of stable networks may not be ensured for the wide range of parameters where $4nc \leq V < 2n(n-1)c$

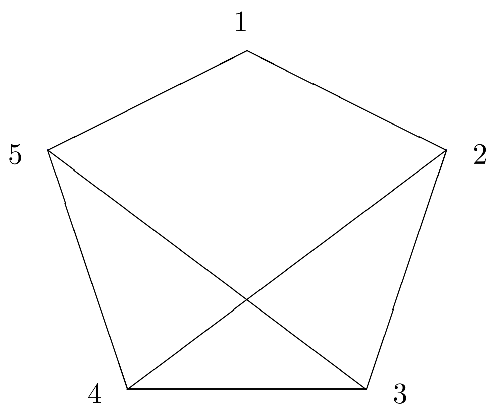
contemplated in Theorem 1. In this section, we shall establish general conditions guaranteeing the existence of symmetric and stable architectures.

Proposition 6 *If n is even or V and c are such that $\eta(V,c)$ is even, pairwise stable symmetric architectures always exist.*

Proof. See the appendix. □

When both n and $\eta(V,c)$ are odd, symmetric stable architectures may fail to exist, as illustrated by the following example.

Example 5 *Let $N = \{1,2,3,4,5\}$ and $V \in [30c, 40c)$. Then $\eta(V,c) = \left\lfloor \frac{V}{2nc} \right\rfloor = 3$. We can check that the only stable architecture with no isolated player is:*



When both n and $\eta(V,c)$ are odd, the existence of stable networks is nonetheless ensured. Moreover, we can characterize a general type of stable architecture that only differs from a symmetric geometry by only one link. We shall refer to this graphs as *almost*-symmetric networks.

Proposition 7 *If n is odd and V and c are such that $\eta(V,c)$ is odd, an almost-symmetric pairwise stable architecture where $n-1$ players have $\eta(V,c)$ neighbors and one player has $\eta(V,c)-1$ neighbors always exists.*

Proof. See the appendix. □

Propositions 6 and 7 guarantee the existence of pairwise stable networks whatever the population size n, the value of the potential bene t V and the value of the link formation cost, thus motivating the following result.

Theorem 2 *Pairwise stable architectures always exist.*

3.5 Efficient networks

Recall that, by de nition, efficient networks are those that maximize the net aggregate bene t.

Proposition 8 *If n is even, the strongly efficient network is a graph consisting of $n/2$ separate pairs. If n is odd, the strongly efficient network is a graph consisting of $(n-3)/2$ separate pairs and 3 players on a line.*[13]

Proof. We easily check that $\sum_{i \in N} \frac{1}{2n}\left[1+\sum_{k \in N(g,i)} \frac{1}{n(g,k)}\right] = 1$. Thus, $\sum_{i \in N} Y_i(g) = V - c\sum_{i \in N} n(g,i)$, implying that maximizing the aggregate payoff in $\mathcal{G}^*$ is equivalent to minimizing $\sum_{i \in N} n(g,i)$ subject to $n(g,i) \geq 1, \forall i \in N$. □

Theorem 1 characterizes the whole set of pairwise stable networks while Corollary 1 focuses on the particular subclass of symmetric stable architectures. Proposition 8 describes the set of efficient networks. Clearly, these two sets do not coincide. The following result clari es this point.

Proposition 9 *Pairwise stable networks are efficient if and only if n is even and $nc \leq V < 2nc$.*

Proof. Indeed, Proposition 8 establishes that efficient networks are such that players have only one neighbor when n is even. Proposition 3 deals with the stability of such networks. □

[13] There are $\frac{n!}{2^{n/2}}$ different possible networks when n is even and $\frac{(n-3)!}{2^{(n-3)/2}}\binom{n}{3}$ when n is odd. When n is even, the set of efficient graphs coincides with that of the co-author model in Jackson and Wolinsky (1996).

Therefore, pairwise stable networks tend to be over-connected from a social viewpoint. This con ict between stability and efficiency is a common feature of the literature on social and economic networks both for directed graphs [e.g. Goyal (1993), Jackson and Wolinsky (1996)] and for undirected networks [e.g. Bala and Goyal (1999), Jackson (1999)]. The reasons for this tension, though, vary with the situations considered. In our context, the intuition is the following. If some player i is the promoter (meaning that she has an idea about a new and pro table economic), she just needs to have at her disposal one potential partner to set up the rm with her and exploit together the gains V from i s idea. But if player i is *not* the promoter (meaning that some other player $j \neq i$ has the idea with value V), she wishes to be connected with as many other players as possible thus increasing the probability of being in direct contact with the promoter j and, consequently, multiplying her chances of being contacted by j to create a rm jointly. Players then establish links as long as they can reasonably bear the cost of a large set of acquaintances. Such a link creation process generates an externality on the set of direct neighbors that is not internalized by any player, and explains the inefficiency of the stable con gurations of links.

References

[1] Aumann, R. and R. Myerson (1988): Endogenous Formation of Links Between Players and Coalitions: An Application of the Shapley Value, in A. Roth ed. *The Shapley Value: Essays in Honour of Lloyd S. Shapley*, Cambridge University Press, 175-191

[2] Bala, V. and S. Goyal (1998): Learning From Neighbors, *Review of Economic Studies* 224, 595-622

[3] Bala, V. and S. Goyal (1999): A Non-Cooperative Theory of Network Formation , forthcoming *Econometrica*

[4] Berge, C. (1970): *Graphes*, Dunod, Paris

[5] Calvó-Armengol, A. (1999a): A Note on Three-Player Noncooperative Bargaining with Restricted Pairwise Meetings , *Economics Letters*, 61, 47-54

[6] Calvó-Armengol, A. (1999b): Bargaining Power in Communication Networks , forthcoming *Mathematical Social Sciences*

[7] Calvó-Armengol, A. (1999c): On Bargaining Partner Selection When Communication is Restricted , manuscript, Universitat Pompeu Fabra and CERAS

[8] Dutta, B. and S. Mutuswami (1997): Stable Networks , *Journal of Economic Theory*, 76, 322-344

[9] Goyal, S. (1993): Sustainable Communication Networks, *Tinbergen Institute Discussion Paper* 93-250, Erasmus University of Rotterdam

[10] Jackson, M. and A. Wolinsky (1996): A Strategic Model of Social and Economic Networks , *Journal of Economic Theory*, 71, 44-74

[11] Jackson, M. and A. Watts (1998): The Evolution of Social and Economic Networks , manuscript, California Institute of Technology and Vanderbilt University

[12] Jackson, M. (1999): The Stability and Efficiency of Directed Communication Networks , manuscript, California Institute of Technology

[13] Kranton, R. and D. Minehart (1998): A Theory of Buyer-Seller Networks , forthcoming *American Economic Review*

[14] Qin, C. (1996): Endogenous Formation of Cooperation Structures, *Journal of Economic Theory* 69, 218-226

[15] Slikker, M. and A. van den Nouweland (1999): Network Formation Models with Costs for Establishing Links, *Research Memorandum FEW* 771, Tilburg University

A Proof of Proposition 1

This proof derives from Proposition 2 and its Corollary in Calvó-Armengol (1999c).[14] The proof is decomposed into the following lemmata:

Lemma 2 *For all $g \in \mathcal{G}^*$ and for all $\langle \tau_g, i \rangle$, where τ_g is a selection on g and $i \in N$, $\exists 0 \leq T \leq n$ and $1 \leq \ell \leq n + 1 - T$ such that $\tau_g^\ell(p_T) = p_T$.*

Proof. Suppose not. Then, for all $0 \leq T \leq n$ and for all $1 \leq \ell \leq n + 1 - T$, $\tau_g^\ell(p_T) \neq p_T$. Fix $T = 0$: by letting ℓ vary between 1 and $n+1$ we obtain that $p_1 = \tau_g^1(p_0), ..., p_{n+1} = \tau_g^{n+1}(p_0)$ are all different of p_0 (where $p_0 = i$). Reiterating the process for all $0 \leq T \leq n$ and with ℓ taking all the values between 1 and $n + 1 - T$, we obtain that $p_0, ..., p_n, p_{n+1}$ are all different two by two, which implies that $|\{p_0, ..., p_{n+1}\}| = n + 1$. Impossible because $p_t \in N$ for all $t \in \mathbb{N}$. □

Lemma 3 (Decomposition) *For all $g \in \mathcal{G}^*$ and for all $\langle \tau_g, i \rangle$, where τ_g is a selection on g and $i \in N$, there exists a unique $0 \leq T^* \leq n$ and a unique $1 \leq \ell^* \leq n + 1 - T^*$ such that the bargaining sequence $\langle \tau_g, i \rangle$ can be decomposed on:*

(a) a nite-horizon bargaining game on a line: nite succession of proposals from p_t to p_{t+1} for $0 \leq t \leq T^* - 1$ where players $p_0, ..., p_{T^*}$ are all different two by two (with $p_0 = i$);

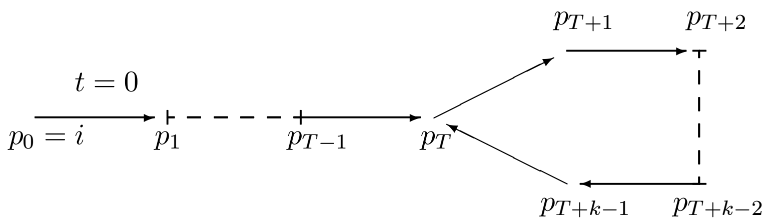
(b) an in nite-horizon bargaining game on a cycle: in nite succession of proposals from p_t to p_{t+1} for $t \geq T^*$, where $p_t = p_{T^*+(t-T^*) \bmod \ell^*}$ and $p_{t+1} = p_{T^*+1+(t-T^*) \bmod \ell^*}$.[15]

Proof. From the previous lemma we know that $\tau_g^\ell(p_T) = p_T$ for some $0 \leq T \leq n$ and $1 \leq \ell \leq n + 1 - T$. Take the smaller T satisfying

[14] These paper addresses the question of bargaining partner selection in a more general setting where the population may be heterogeneous in time preferences −players have different discount rates.

[15] In other words, the in nite sequence of proposers and respondents from round T on is: $p_T, p_{T+1}, \ldots, p_{T+\ell-1}, p_T; p_{T+1}, \ldots, p_{T+\ell-1}, p_T, \ldots$

this property and the corresponding ℓ, denoted respectively by T^* and ℓ^*. Clearly $p_0, ..., p_{T^*}$ are all different two by two because otherwise T^* is not the smaller T satisfying the property of the previous lemma. We also know that $\tau_g^{\ell^*}(p_{T^*}) = p_{T^*}$. Let $t \geq T^*$: $t = T^* + k\ell^* + (t - T^*) \bmod \ell^*$ for some $k \in \mathbb{N}$. Then $p_t = p_{T^*+k\ell^*+(t-T^*) \bmod \ell^*} = \tau_g^{k\ell^*}\left(p_{T^*+(t-T^*) \bmod \ell^*}\right) = p_{T^*+(t-T^*) \bmod \ell^*}$. □



We analyze the subgame perfect equilibria of this two type of games. Let g be a connected graph on N, and let $\langle \tau_g, i \rangle$ be a bargaining sequence, where τ_g is a selection on g and $i \in N$. Let T and ℓ given by the Decomposition Lemma: T is the length of the nite-horizon bargaining game from round $t = 0$ until round $t = T - 1$, and ℓ is the length of the cycle supporting the in nite-horizon bargaining game beginning at round $t = T$. Note that necessarily $\ell \geq 2$. Indeed, ℓ is such that $\tau_g^{\ell}(p_T) = p_T$ and by de nition of the selection function $\tau_g(p_T) \subset N(g, p_T) \subset N \backslash \{p_T\}$, implying that $\tau_g(p_T) \neq p_T$. Note also that when $\ell = 2$, the in nite-horizon bargaining game is a standard 2−player alternating offers game between the players labeled p_T and p_{T+1}, beginning at round $t = T$.

We introduce some useful notations. For any two rounds t and s ($t \geq T$, $s > 0$), $\omega^{\tau_g}(t, t+s)$ denotes the product of the discount factors of the players that respond to offers from round t to round $t+s-1$ (or, equivalently, that make offers from round $t+1$ to round $t+s$):

$$\omega^{\tau_g}(t, t+s) = \delta_{p_{t+1}} \times \cdots \times \delta_{p_{t+s}} = \prod_{i=1}^{s} \delta_{p_{t+i}}.$$

Let also $a_t(g) = \frac{1}{1-\omega^{\tau_g}(0,2\ell)} \left[1 + \sum_{i=1}^{2\ell-1} (-1)^i \omega^{\tau_g}(t, t+i)\right]$, for all $t \geq T$. Finally, denote by $\Delta_n = \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 1; x_1, \ldots, x_n \geq 0\}$ the unit simplex of $\mathbb{R}_+^n$.

Lemma 4 *The in nite-horizon bargaining game on the cycle of length ℓ beginning at round T has a unique perfect equilibrium outcome supported by the following set of stationary strategies, where offers made at any stage correspond to the perfect equilibrium outcome and players agree to any offer of at least this amount: at round $t \geq T$, player $p_t (= p_{T+(t-T) \bmod \ell})$ proposes the share $1 - a_t(g)$ to player p_{t+1} and accepts any offer $x \in \Delta_n$ from player p_{t-1} as long as $x_{p_t} \geq a_t(g)\delta_{p_t}$.*

Proof. It is easy to check that the strategies described above are a subgame perfect equilibrium. To establish uniqueness, we show that the in mum and the supremum of the set of equilibria payoffs coincide. For all $t \geq T$, let $\mathfrak{S}_t$ be the set of perfect equilibrium payoffs of player p_t at period t for the subgame where p_t is the initiator of the bargaining procedure (subgame beginning at round t). Let $S_t = \sup \mathfrak{S}_t$ and $s_t = \inf \mathfrak{S}_t$, $\forall t = 0, 1, \ldots$. The proof is in four steps.[16]

Step 1. $S_t \leq 1 - \delta_{p_{t+1}} s_{t+1}$, $\forall t \geq T$

Proof. At round $t+1$, the proposer p_{t+1} can get at least s_{t+1} which corresponds to a discounted payoff of $\delta_{p_{t+1}} s_{t+1}$ at round t. Hence, at round t the proposer p_t can not offer less than $\delta_{p_{t+1}} s_{t+1}$ to her bargaining partner (respondent) p_{t+1} for the offer to be accepted. Therefore, at round t the proposer p_t can not get more than a share of $1 - \delta_{p_{t+1}} s_{t+1}$, that is $S_t \leq 1 - \delta_{p_{t+1}} s_{t+1}$. *Q.E.D.*

Step 2. $s_t \geq 1 - \delta_{p_{t+1}} S_{t+1}$, $\forall t \geq T$

Proof. Similar argument than in the previous step. *Q.E.D.*

Step 3. $S_t \leq \frac{1}{1-\omega^{\tau_g}(0,2\ell)} \left[1 + \sum_{i=1}^{2\ell-1} (-1)^i \omega^{\tau_g}(t, t+i)\right]$, $\forall t \geq T$

Proof. Let $t \geq T$. We show by induction on k that:

$$S_t \leq 1 + \sum_{i=1}^{2k-1} (-1)^i \prod_{s=1}^{i} \delta_{p_{t+s}} + \prod_{s=1}^{2k} \delta_{p_{t+s}} \cdot S_{t+2k}, \forall k = 1, 2, \ldots \quad (1)$$

Combining the results of the two previous steps we get:

[16] This proof follows a similar argument than that of Proposition 1 in Calvó-Armengol (1999a).

$$S_t \leq 1-\delta_{p_{t+1}}\left(1-\delta_{p_{t+2}}S_{t+2}\right) \Leftrightarrow S_t \leq 1-\delta_{p_{t+1}}+\delta_{p_{t+1}}\delta_{p_{t+2}}S_{t+2}$$

Hence, the result is true for $k=1$. Suppose then that it is true for some $k \geq 1$. We show that it still holds at $k+1$. Indeed, combining again the results of steps 1 and 2, $S_{t+2k} \leq 1-\delta_{p_{t+2k+1}}+\delta_{p_{t+2k+1}}\delta_{p_{t+2(k+1)}}S_{t+2(k+1)}$. We then obtain a new upper bound for S_t:

$$\begin{gathered} S_t \leq 1+\sum_{i=1}^{2k-1}(-1)^i\prod_{s=1}^{i}\delta_{p_{t+s}}+ \\ \prod_{s=1}^{2k}\delta_{p_{t+s}}\left(1-\delta_{p_{t+2k+1}}+\delta_{p_{t+2k+1}}\delta_{p_{t+2(k+1)}}S_{t+2(k+1)}\right) \\ \Leftrightarrow S_t \leq 1+\sum_{i=1}^{2k+1}(-1)^i\prod_{s=1}^{i}\delta_{p_{t+s}}+\prod_{s=1}^{2(k+1)}\delta_{p_{t+s}}\cdot S_{t+2(k+1)} \end{gathered}$$

Hence, if the inequality is true at k it still holds at $k+1$. Therefore, by induction, the inequality (1) holds for all integer $k \geq 1$, and for all integer $t \geq T$. We know from the description of the bargaining procedure that for all $t \geq T$, $p_{t+\ell}=p_t$. Indeed, after ℓ rounds all the players on the cycle have been successively a proposer and a responder, and a new complete tour of bilateral bargaining contests begins. Also, for all $t \geq T$, $p_{t+2\ell}=p_t$.[17] Therefore, $S_{t+2\ell}=S_t$, $\forall t \geq T$. Then, if we let $k=\ell$ in (1) we obtain:

$$S_t\left(1-\prod_{s=1}^{2\ell}\delta_{p_{t+s}}\right) \leq 1+\sum_{i=1}^{2\ell-1}(-1)^i\prod_{s=1}^{i}\delta_{p_{t+s}}$$

And with the notation introduced at the beginning of this section:

$S_t \leq \frac{1}{1-\omega^{\tau_g}(0,2\ell)}\left[1+\sum_{i=1}^{2\ell-1}(-1)^i\omega^{\tau_g}(t,t+i)\right]$, $\forall t \geq T$. *Q.E.D.*

Step 4. $s_t = S_t = \frac{1}{1-\omega^{\tau_g}(0,2\ell)}\left[1+\sum_{i=1}^{2\ell-1}(-1)^i\omega^{\tau_g}(t,t+i)\right]$, $\forall t \geq T$.

Proof. With a similar argument than in the previous step we show that:

$s_t \geq \frac{1}{1-\omega^{\tau_g}(0,2\ell)}\left[1+\sum_{i=1}^{2\ell-1}(-1)^i\omega^{\tau_g}(t,t+i)\right]$, $\forall t \geq T$. *Q.E.D.*

The result follows immediately as, by de nition, $s_t \leq S_t$, $\forall t \geq T$. □

[17] If ℓ is even, we do not need to make two complete tours of the cycle ($k=\ell$). One such tour (i.e. $k=\ell/2$) suffices and we end up with the reduced expression $a_t(g) = \frac{1}{1-\omega^{\tau_g}(0,\ell)}\left[1+\sum_{i=1}^{\ell-1}(-1)^i\omega^{\tau_g}(t,t+i)\right]$, for all $t \geq T$.

Lemma 5 *When the population is homogeneous in time preferences, at the unique perfect equilibrium of the in nite-horizon bargaining game on a cycle, all players make the same standard cake division proposal* $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ *to their bargaining partner.*

Proof. For all $t \geq T$ and $s > 0$, $\omega^{\tau_g}(t, t+s) = \delta^s$. Therefore, $a_t(g) = \frac{1-\delta+\cdots-\delta^{2\ell-1}}{1-\delta^{2\ell}} = \frac{1}{1+\delta}$. □

Lemma 6 *When the population is homogeneous in time preferences, at the unique perfect equilibrium of the nite-horizon bargaining game on a line, all players make the same standard cake division proposal* $\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ *to their bargaining partner.*

Proof. The player labeled p_T can ensure $\frac{1}{1+\delta}$ at the round T by initiating the in nite-horizon bargaining game on the cycle. Therefore, at round $T-1$ the player labeled p_{T-1} offers p_T a share $1 - a_{T-1} = \frac{\delta}{1+\delta} \Leftrightarrow a_{T-1} = \frac{1}{1+\delta}$. The proof then follows by backward induction. □

B Existence of stable networks: proofs

The proofs of existence are an application of the following general existence theorem on graph theory due to Erdös and Gallai (1960) [see for instance Berge (1970)]:

Theorem 3 *Let* $d_1 \geq d_2 \geq \cdots \geq d_n$ *a sequence of* n *integers such that* $\sum_{i=1}^{n} d_i$ *even. The two following conditions are equivalent:*

(a) $\exists g \in \mathcal{G}$ *such that* $n(g,i) = d_i$, $\forall i \in N$*;*

(b) $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}$, $\forall k \in \{1, \ldots, n\}$*.*

We use this theorem to prove Propositions 6 and 7.

Proof of Proposition 6. For notational simplicity, let $\eta \equiv \eta(V,c)$. Let $d_1 = \cdots = d_n = \eta(V,c)$. If n is even or if η is even, $\sum_{i=1}^{n} d_i = n\eta$ is also even. Proving that a pairwise stable symmetric architecture with size η

exists is thus equivalent to showing that condition (b) of Theorem 3 holds. This condition is equivalent to:

$$k\eta \leq k(k-1) + (\eta - k - 1)k + (n - \eta + 1)\eta, \forall k \in \{1, \ldots, \eta - 2\} \quad (2)$$

$$k\eta \leq k(k-1) + (n-k)\eta, \forall k \in \{\eta - 1, \ldots, n\} \quad (3)$$

Condition (2) is equivalent to $2k \leq \eta(n - \eta + 1), \forall k \in \{1, \ldots, \eta - 2\}$. But as $k \leq \eta$, it suffices to show that $2\eta \leq \eta(n - \eta + 1) \Leftrightarrow \eta \leq n - 1$ which is true in the cases considered where $V < 2n(n-1)c$.

Let $u_k = k(k-1) + (n-k)\eta - k\eta$. Condition (3) is equivalent to $u_k \geq 0, \forall k \in \{\eta - 1, \ldots, n\}$. Let $f(x) = x(x-1) + (n-x)\eta - x\eta$ de ned on $\mathbb{R}$: $f'(x) = 2x - 1 - 2\eta$ and $f''(x) = 2$. Thus f is strictly convex and reaches its (global) minimum at $x^* = \eta + \frac{1}{2}$. Hence,

$$\min_{k \in \{\eta - 1, \ldots, n\}} u_k = \min\{u_\eta, u_{\eta+1}\} = \eta(n - 1 - \eta) \geq 0$$

for the cases considered where $V < 2n(n-1)c$.

An alternative constructive proof of this proposition when $\eta(V,c)$ is even is the following. Locate the players on a circle. For each player establish $\frac{\eta(V,c)}{2}$ links with the $\frac{\eta(V,c)}{2}$ closest players on the circle at her right-hand side and $\frac{\eta(V,c)}{2}$ links with the $\frac{\eta(V,c)}{2}$ closest players at her left-hand side. A sufficient condition for such a network to be able to constructed is that there are enough players that is, $1 + 2\frac{\eta(V,c)}{2} \leq n \Leftrightarrow \eta(V,c) \leq n - 1$ which again is true in the cases considered where $V < 2n(n-1)c$. □

Proof of Proposition 7. Let $d_1 = \cdots = d_{n-1} = \eta$ and $d_n = \eta - 1$. If n and η are odd, $\sum_{i=1}^{n} d_i = (n-1)\eta + \eta - 1$ is even. Proving that a pairwise stable architecture with neighborhood sizes d_i exists is thus equivalent to showing that condition (b) of Theorem 3 holds. This condition is equivalent to:

$$k\eta \leq k(k-1) + (\eta - k - 1)k + (n - \eta + 1)\eta, \forall k \in \{1, \ldots, \eta - 2\} \quad (4)$$

$$k\eta \leq k\,(k-1) + (n-k-1)\,\eta + \eta - 1, \forall k \in \{\eta - 1, \ldots, n-1\} \tag{5}$$

$$(n-1)\eta + \eta - 1 \leq n\,(n-1) \tag{6}$$

Condition (4) coincides with condition (2) already established in the previous proof. Condition (6) is equivalent to $n(n-1-\eta) \geq -1$ which is true. We are thus left with condition (5).

Let $u_k = k\,(k-1) + (n-k)\,\eta - k\eta$. Condition (5) is then equivalent to (after some algebra) $\min_{k\in\{\eta-1,\ldots,n-1\}} u_k = \eta(n-1-\eta) \geq 1$. The complete graph g^N is such that $\eta = n-1$. We are assuming throughout that $V < 2n(n-1)c$. Therefore, $\eta < n-1$ and condition (5) is equivalent to $\eta \geq 1$ which is true because η is odd. □