# Equilibrium Reserve Prices in Sequential Ascending Auctions*

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# Abstract

We study a model in which the same set of bidders, with perfectly correlated valuations across units, compete for two units of a good in two sequential ascending-price auctions. The seller sets a reserve price before the beginning of each auction. Surprisingly, the equilibrium has a simple structure; strategic non-disclosure of information (i.e., pooling) only takes the form of non-participation in the early auction by bidders with valuations below a threshold, while bidders with valuations above the threshold participate and bid truthfully; that is, they stay in the auction until the price reaches their true valuations. The participation threshold is strictly higher than the reserve price in the first auction, so some buyers who would find it profitable to buy at the reserve price select not to participate in order to attempt to decrease the reserve price in the second auction. Participation in the first auction is lower than under full commitment, but the probability of at least one bidder participating in the second auctions is higher. As a by-product, our analysis provides a rigourous argument in favor of the theoretical advantages of ascending auctions.

**Keywords:** repeated auctions, ratchet effect, participation, reserve price.

**JEL:** D44 & D82.

# 1 Introduction

In recent years there has been a surge of theoretical, empirical and practical interest in auctions (see, for example, Klemperer's survey (1999) and Milgrom's book (2001)). A great deal of attention has been devoted to the simultaneous ascending auction of multiple units to bidders with multi-unit demands due, at least in part, to the successful use of such auctions by the FCC to allocate spectrum rights. A consensus has emerged that this auction format, with or without combinatorial bids, is particularly appropriate for designing procedures to allocate the rights to operate in newly created markets.

One of the important arguments, besides those related to the Linkage Principle,[1] in favor of an ascending auction as opposed to a first-price or a Vickrey auction, is that bidders are reluctant to reveal their true valuations. Rothkopft et al. (1990, pp. 108) first note: "Vickrey auctions are rare because bidders are reluctant to follow the truth-revealing strategies that the 'proper' operation of such auctions would require. Bidders have good reasons to be reluctant when they may lose a fraction of the economic rent revealed by the sealed second-price format in subsequent negotiations."[2] If revealing one's valuation leads to potential future losses, then equilibrium in a first-price and a Vickrey auction involves substantial pooling of bids and complex equilibrium strategies. At work here is the well-known ratchet effect from the planning and principal-agent literature. Laffont and Tirole (1988), for example, show that when information takes a continuum of values, local dynamic incentive compatibility implies that separation of agent types is not feasible. A lot of pooling occurs in equilibrium; that is, agents undertake identical actions for different values of their information.

In what precise sense are ascending auctions theoretically superior on that respect? Since a bidder's

[1] See in particular the seminal paper by Milgrom and Weber (1982).

[2] Ausubel and Milgrom (2001, pp. 6-7) elaborate on this: "the revelation of bidders' maximum willingness to pay during the auction can be problematic ... Winning bidders may fear that information revealed by their bids will be used by auctioneers to cheat them or by third parties to disadvantage them in some negotiation ... A bidder's motive to conceal its information can destroy the dominant strategy property that accounts for much of the appeal of the Vickrey auction ... a similar case can be made against ordinary first-price auctions, since the theoretical bid functions are invertible to reveal bidders' values. In this respect, ascending auctions are theoretically superior to both kinds of sealed bid auctions because they better conceal the winning bidder's valuation."

decision to participate and stay in the auction does reveal some information to the seller, it is far from obvious that little pooling and a simple equilibrium structure would emerge in ascending auctions. In this paper, we provide a rigorous answer to the previous question: we prove that ascending auctions are immune, or at least less prone, than first-price or Vickrey auctions to ratcheting effects in a dynamic setting. The setting we analyze is one in which bidders are justified in their reluctance to reveal their valuations: we study sequential auctions, in which bidders have correlated valuations for multiple units, while the seller can freely set the reserve price at the beginning of each auction, which enables him to exploit in late auctions information about bidders that might have filtered out in early auctions.

The setting is also one of clear practical importance. Many goods, services and contracts are allocated in sequential auctions, sometimes with quite long time periods between two consecutive auctions, sometimes with several auctions almost in a row. As documented in the literature, estate (art, stamps, books, etc.), cattle, fish, vegetables, timber and wine are often allocated in comparable lots at sequential auctions, to a quite well-established and limited group of potential buyers.[3]

There are several reasons that explain the use of sequential auction procedures in practice. First, the units may not be available at the same time and yet, when available, they may be perishable and would then have to be sold separately. This is a particularly relevant explanation for fish, flowers, timber, vegetables, etc.[4] Second, when the auctioning party is a government or a public authority, committing to a future reserve price, or pricing policy, is often impossible or illegal. This is relevant for the auction of many kinds of concession contracts, operating licenses and leases in which the duration of the contract is of several years, and it is anticipated that another auction will take place at the renewal stage. In spite of their importance, these auctions have not been studied in the literature; this paper fills this gap.

We assume that bidders' valuations for the units, in two sequential auctions, are perfectly correlated

[3] See, e.g., Cassady (1967), Ashenfelter (1989), Donald, Paarsch and Robert (1997, 2001), Laffont, Ossard and Vuong (1995) and Milgrom (2001).

[4] See, e.g., the discussion in Weber (1983) and, for details concerning timber auctions, Donald, Paarsch and Robert (2001).

across periods, since strategic retention of information is particularly severe under these conditions. Nevertheless, we are able to characterize the equilibrium path and to show that in our model strategic non-disclosure of information (i.e., pooling) only takes the form of non-participation in the early auction. Bidders with valuations below a threshold do not participate in the first auction, while bidders with valuations above the threshold participate and bid truthfully; that is, they stay in the auction up to the point where the price reaches their true valuation. Intuitively, this is because in an ascending auction the winner only reveals a lower bound on his valuation. The participation threshold is strictly higher than the reserve price in the first auction, so some buyers who would find it profitable to buy at the reserve price select not to participate in order to attempt to decrease the reserve price in the second auction.

If it were possible, the seller would want to commit to the same reserve price in both auctions. This reserve corresponds to the reserve price in the optimal static auction for a single item. We show that in our model participation in the first auction is lower than under full commitment, but that the probability of at least one bidder participating in the second auction is higher than in the optimal auction. Finally, we show that our results are specifically due to the ascending auction format since, in a sequence of two second-price auctions, other things being identical, the ratchet effect prevents information disclosure in the early auction and induces pooling.

There are some papers that study the choice of a reserve price in *single-unit* auctions but with some dynamic considerations. McAfee and Vincent (1997) study single-item auctions in which the seller can set a reserve price, but cannot commit not to re-auction the object if a sale fails. They prove the revenue equivalence of a first and a second-price auction. They also show that if the time between auctions goes to zero, then the seller's expected revenue converges to the revenue of a static auction without a reserve price. Burguet and Sakovics (1996) study a single-item, first-price auction, with a large number of potential bidders and endogenous entry. Bidders who decide to participate must pay a cost to learn their valuations; the seller may set a reserve price, but if the object goes unsold, he must offer it for sale at another auction without a reserve price. They show that a reserve price that restricts participation may be optimal. Haile (2000) studies a single-item, second-price auction with a reserve price between two bidders with imperfect information about their valuations. If the good is sold, then all bidders' valuations become known and efficient bargaining takes place if there are gains to be realized (e.g., the item is not in the hands of the highest valuation bidder). Haile shows that in equilibrium there is partial pooling of bids at the reserve price.

Among the papers on sequential auctions with multi-unit demands, the first to be mentioned is Ortega Reichert's (1968) analysis of a two-bidder, two-period, sequential first-price sealed bid auction with positive correlation of bidders' valuations across periods and across bidders. He shows that a bidder in the first auction is less aggressive than in a one-shot auction, so as to induce more pessimistic beliefs by his rivals about his likely future valuation and thus reduce competition in the second auction. Hausch (1986) studies a similar model, but focuses on whether the seller prefers to sell the two objects sequentially or simultaneously. The underbidding uncovered by Ortega Reichert has a negative effect on seller's revenue. Hausch shows that there is an opposite, informational effect at work in a sequential auction that raises the seller's revenue; more information may be revealed.

Weber (1983) is a standard reference for the discussion of a number of issues relating to sequential auctions, including bidders' multi-unit demands. More recently, Donald, Paarsch and Robert (1997, 2001), Montmarquette and Robert (1999), and Katzman (1999) analyze sequential auctions where bidders are interested in acquiring several units, but they focus on models where bidders' valuations for the different units are independent (but ranked in decreasing order). Therefore, in these models, the bidder who wins a given auction is not concerned about revealing his valuation. Luton and McAfee (1986) study a sequential procurement auction in which the auctioneer (the buyer in their case) can commit to an optimal mechanism. In their model bidders learn between auctions, and a bidder's cost in the second auction may be lower (but never higher) than in the first. They show that the buyer commits to a policy that discriminates in the second auction against the winner of the first.[5]

This paper proceeds as follows. In the next section we introduce the model. In Section 3 we study the continuation equilibria for a fixed, first-auction reserve price. In Section 4 we complete the characterization of the equilibrium of our sequential ascending auction. In Section 5 we show that equilibrium in a sequential second-price auction exhibits a lot of pooling; the first-auction bids never reveal the bidders' valuations. Section 6 concludes.

# 2 The Model

## 2.1 Information and auction format

A seller wants to sell two units of a good to $n$ potential bidders, indexed by $i$, in a sequence of two ascending-price auctions. While being committed to the use of ascending price auctions, at the start of each auction the seller is free to choose a reserve price, below which he will not sell the good. His valuation for each unit of the good is constant and normalized to 0. Let $\delta \in (0,1]$ denote the discount factor common to all bidders and the seller.

Bidder $i$'s valuation for each unit of the good is $v_i$; that is, a bidder's valuations for the units are perfectly correlated. Bidders' valuations are private information and are identically and independently distributed across bidders, according to the c.d.f. $F(\cdot)$, with positive density $f(\cdot)$, over $[\underline{v}, \overline{v}]$. Let $G(\cdot)$ (respectively $g(\cdot)$) denote the c.d.f. (density) of the highest order statistics $v_{-i}^{(1)} = \sup\{v_1, ...v_{i-1}, v_{i+1}, ....v_n\}$ for a given $i$: $G(v) = [F(v)]^{n-1}$ .Let a bidder's virtual valuation function be given by

$$J(v) \equiv v - \frac{1 - F(v)}{f(v)}.$$

[5] Several papers are concerned with the so-called "declining price anomaly," which relates to evidence that in sequential auctions, sale prices have a tendency to decrease: see e.g. Ashenfelter (1989), Black and deMeza (1992), Gandal (1997), McAfee and Vincent (1993), Gale and Hausch (1994), Gale and Stegeman (2001), von-der-Fehr and Morch (1994), Jeitschko (1999) and Gale, and Hausch and Stegeman (2000). Contrary to our paper, they do not allow for an active role for the seller.

We will make the standard regularity assumption of double monotone likelihood ratio property:

**Assumption:** $\frac{F(v)}{f(v)}$ is increasing and $\frac{1-F(v)}{f(v)}$ is decreasing in $v$.

The seller sets the reserve price $R_1$ for the first ascending-price auction and then, after having observed the first-auction outcome, the reserve price $R_2$ for the second auction. In each auction, the price starts at the reserve price. If all bidders are "out", there is no sale. If only one bidder is "in", he gets the good at the reserve price. If at least two biders are "in", the price continuously increases until all bidders, except one, leave the process. The winner pays the highest price at which his most aggressive rival got out. All ties are resolved by random draws.[6]

We study the symmetric perfect Bayesian equilibria of this game with the following standard properties: (1) bidders only use (weakly) undominated pure strategies; (2) each bidder's bid function is piecewise (weakly) monotone; that is, it has at most a finite number of discontinuities, with one-sided limits at each point of discontinuity, and $[\underline{v}, \overline{v}]$ is partitioned into a finite number of subintervals over each of which the function is weakly monotone.[7]

## 2.2 Benchmarks

The optimal static auction for a single object in a one-shot version of this setting has been characterized by Myerson (1981). In this symmetric and regular setting, the revenue equivalence theorem applies and the optimal (revenue-maximizing) auction procedure can be implemented by any of several standard auctions (e.g., first-price, second-price, ascending, etc.) with a reserve price $R_0$ given by

$$R_0 = \begin{cases} J^{-1}(0) & \text{if } J(\underline{v}) < 0 \\ \underline{v} & \text{otherwise.} \end{cases}$$

When he gets ex ante the additional information that all bidders have valuations in a sub-interval $[v_L, v_H] \subset [\underline{v}, \overline{v}]$, the seller's updates his beliefs so that bidders' valuations are i.i.d. according to the c.d.f.

[6] This version is often called a Japanese auction.

[7] As we shall see, contrary to the static case, in our setting monotonicity of the bid function over the whole range $[\underline{v}, \overline{v}]$ does not immediately follow from incentive compatibility.

$\frac{F(\cdot)-F(v_L)}{F(v_H)-F(v_L)}$ on $[v_L, v_H]$. Let

$$J_{v_L,v_H}(R) \equiv R - \frac{F(v_H) - F(R)}{f(R)}.$$

It is simple to check that under our assumption on $F(\cdot)$, $J_{v_L,v_H}(R)$ is an increasing function of $R$ and a decreasing function of $v_H$; hence, the optimal reserve price $R_{v_L,v_H}$ is determined by:

$$R_{v_L,v_H} = \begin{cases} J^{-1}_{v_L,v_H}(0) & \text{if } J_{v_L,v_H}(v_L) < 0 \\ v_L & \text{otherwise.} \end{cases}$$

$R_{v_L,v_H}$ is non-decreasing in $v_H$ and in $v_L$ (in fact, if $J_{v_L,v_H}(v_L) \geq 0$, then $R_{v_L,v_H}$ does not depend on $v_L$). Note also that this optimal reserve price does not depend on the number of bidders.

In our repeated auction setting, suppose that the seller can commit to the sequence of reserve prices $R_1$ and $R_2$ in the successive auctions (with $R_2$ possibly depending on the outcome of the first auction). By the revelation principle, the seller cannot raise a higher revenue than the one he would obtain in an optimal static mechanism in which buyers are asked to report their valuations. Since buyers have the same valuation for the item in each period, the optimal mechanism for the seller is Myerson's optimal auction of the bundle composed by the two units of the good, with a reserve price of $R_0 + \delta R_0$. It is clear that if the seller must sell the goods in sequence, but can commit to a sequence of reserve prices, then Myerson's optimal auction can be implemented by a repeated standard auction with a reserve price of $R_0$ in both periods.

The problem is, of course, that after the conclusion of the first auction the seller would want to renege on his promise not to change the reserve price. Moreover, if no bidder participated in the first auction both the seller and the bidders would prefer to break the seller's commitment and renegotiate.

# 3 Bidding Behavior for a Given First Auction Reserve Price

From now on, we suppose that the seller cannot commit to the sequence of reserve prices. In this section, moreover, we take the first auction reserve price $R_1$ as given and investigate the continuation equilibrium.

The second auction is a standard ascending auction with private values. So, irrespective of their beliefs, it is a weakly dominant strategy for bidders to participate as long as their valuation exceeds the reserve price $R_2$, and to exit when the price reaches their true valuation $v_i$. The second auction optimal reserve price, however, depends on the seller's beliefs about the bidders' valuations; each bidder has then an incentice to deviate from the one-shot truthful bidding strategy in the first ascending auction in order to try to induce the seller to set a lower reserve price in the second auction. It is therefore critical to analyze the amount of information that filters out of the first auction. The seller knows at what price each loser dropped out, who won the auction, and what was the final price reached in the process. How these data are interpreted depends upon equilibrium bidding behavior in the first auction.

## 3.1 Semi-separation with participation in the first auction

We begin by looking at the case in which a positive measure of bidder's types participate in the first auction. As we pointed out in the introduction, this setting exhibits strong similarities with dynamic contracting models without commitment, such as Laffont and Tirole (1988). In their model no characterization of the continuation equilibrium is available. It is only known that full separation of types is impossible and a lot of pooling occurs in equilibrium. As will appear below, a remarkable result in our setting is that equilibrium has a simple structure and pooling only takes the form of strategic non-participation.

In static auction settings, a standard revealed preference argument shows that bidding functions must be at least weakly monotone increasing in bidders' types. Basically, this property is a direct consequence of the property that bidders' objective functions exhibit single crossing. No such direct conclusion is available in our setting. In our dynamic model, the seller's beliefs at the beginning of the second auction are endogenous. An increase in first-auction bidding may trigger an increase or a decrease in the equilibrium reserve price in the second auction, and hence a decrease or an increase in the probability of getting the second unit and in its equilibrium price for the winner. In other words, the single-crossing property of the expected payoffs of bidders at the beginning of the first auction is an endogenous equilibrium property that has to be proved along with the characterization of equilibrium behavior. Lemma 1 and 2 below characterize the equilibrium bidding functions and show that they are indeed monotone increasing. Although the result is natural, the intricacy of the proof shows that the intuition simply based on single-crossing arguments is not sufficient in complicated (dynamic) strategic settings.

Given the reserve price $R_1$, suppose that a symmetric, pure strategy equilibrium of this continuation game exists, and let $\beta(\cdot)$ be the first-auction equilibrium bidding function. We will restrict attention, w.l.o.g., to right-continuous bidding functions.[8] This enables us to define $v_*$ as the lowest type that participates in the first auction. Suppose $v_* < \overline{v}$; that is, suppose a positive measure of types participates in the first auction. Piecewise monotonicity of $\beta(\cdot)$ implies that there is an interval of types $[v_*, v_T)$ where the function $\beta(\cdot)$ is continuous. The point $v_T$ is the first point of discontinuity of $\beta(\cdot)$ to the right of $v_*$ (by definition, types below $v_*$ do not participate in the first auction). If $\beta(\cdot)$ is everywhere continuous to the right of $v_*$, we let $v_T$ equal $\overline{v}$. We begin by showing that types in the interval $[v_*, v_T)$ bid their true valuations.

**Lemma 1** *: Suppose the continuation game following a given $R_1$ has a symmetric, pure strategy equilibrium and that a positive measure of types participates in the first auction. Let $v_*$ be the lowest type participating in the first-period auction and $v_T$ be the first point of discontinuity of $\beta(\cdot)$ to the right of $v_*$. Then, $\beta(v) = v$ for all $v \in [v_*, v_T)$.*

**Proof.** When there is active bidding in the first auction, it becomes known to the seller that the

[8] If $v_T$ is a point of discontinuity of $\beta(\cdot)$, then it must be the case that type $v_T$ is indifferent between bidding $\liminf_{v\to v_T} \beta(v)$ and $\limsup_{v\to v_T} \beta(v)$. This follows from the fact that, for all $\varepsilon > 0$, type $v_T + \varepsilon$ prefers the bid $\beta(v_T + \varepsilon)$ to the bid $\beta(v_T - \varepsilon)$ while type $v_T - \varepsilon$ has the opposite preference. The restriction is therefore innocuous.

highest valuation is at least equal to $v_*$. Therefore $R_2 \geq v_*$ and type $v_*$ cannot make any profit in the second auction. Type $v_*$ then bids in the first auction as in a static auction: $\beta(v_*) = v_*$ (which imposes $v_* \geq R_1$).

Fix any $v_0$ such that $v_* \leq v_0 < v_T$ and suppose that $\beta(v_0) = v_0$ while for any $v < v_0$, $\beta(v) < v_0$ (and we let $\beta(v) = 0$ for a non-participating type $v < v_*$). We proceed in several steps to prove that to the right of $v_0$, it must be the case that $\beta(v) = v$ as well. Then, given that $\beta(\cdot)$ is continuous in $[v_*, v_T)$, the lemma follows.

**Step 1:** $\beta(\cdot)$ cannot be locally constant on some $[v_0, v_0 + \varepsilon_0)$.

Suppose, to the contrary, that $\beta(v_0 + \varepsilon) = v_0$ for all $\varepsilon \in [0, \varepsilon_0)$. Consider bidder $i$ of type $v_0 + \varepsilon$ deviating to $v_0 + \varepsilon$ instead of bidding $v_0$. The first-auction gains from this deviation are at least equal to the gains from being the sole winner instead of having at best a $\frac{1}{2}$ probability of winning when $v_{-i}^{(1)} \in [v_0, v_0 + \varepsilon_0)$, that is:

$$\frac{\varepsilon}{2} \left[ G(v_0 + \varepsilon_0) - G(v_0) \right]. \tag{1}$$

This deviation may also cause a loss in the second auction. This can only happen in the event that the deviation affects (increases) $R_2$ and the bidder expects positive gains from the second auction, that is when $v_{-i}^{(1)} \in [v_0, v_0 + \varepsilon)$. At worst, in this event, the deviation induces $R_2 \geq v_0 + \varepsilon$ while bidding $\beta(\cdot)$ would induce $R_2 \leq v_0$; that is, an upper bound of the loss is:

$$\delta \int_{v_0}^{v_0+\varepsilon} (v_0 + \varepsilon - y) \, dG(y). \tag{2}$$

The net expected gain from the deviation is bounded from below by the difference between (1) and (2), that we call $\Delta(\varepsilon)$. It is simple to see that $\Delta(0) = 0$ and $\Delta'(0) > 0$. Hence for a sufficiently small $\varepsilon$, $\Delta(\varepsilon) > 0$ and the deviation is profitable.

**Step 2:** $\beta(\cdot)$ cannot be locally decreasing on some $[v_0, v_0 + \varepsilon_0)$.

Suppose, to the contrary, that $\beta(\cdot)$ is strictly decreasing on $[v_0, v_0 + \varepsilon_0)$. Then, the analysis of the deviation by type $v_0 + \varepsilon$ from $\beta(v_0 + \varepsilon)$ to $v_0 + \varepsilon$ proceeds along similar lines as in Step 1 and yields a lower bound for the expected gains equal to:

$$\Delta(\varepsilon) = \int_{v_0}^{v_0+\varepsilon} (v_0 + \varepsilon - \beta(y))\, dG(y) - \delta \int_{v_0}^{v_0+\varepsilon} (v_0 + \varepsilon - y)\, dG(y)$$

which is positive for small enough $\varepsilon > 0$.

**Step 3:** $\beta(\cdot)$ cannot be strictly increasing and such that $\beta(v) < v$ on some interval $(v_0, v_0 + \varepsilon_0)$.

Suppose to the contrary that, for all $\varepsilon \in [0, \varepsilon_0)$, $\beta(\cdot)$ is strictly increasing and $\beta(v_0 + \varepsilon) < v_0 + \varepsilon$. A deviation by bidder $i$ of type $v_0 + \varepsilon$ from $\beta(v_0 + \varepsilon)$ to $v_0 + \varepsilon$ strictly increases his first auction gains since there is a positive probability that the rival has valuation in $(v_0 + \varepsilon, v_0 + \varepsilon_0)$. This deviation can only make a difference in the second auction if $v_{-i}^{(1)} < v_0 + \varepsilon$, otherwise bidder $i$ does not expect any profit from the second auction. But this implies that rivals drop out at the same prices, irrespective of $i$'s deviation, and $R_2$ cannot be affected by $i$'s deviation. The deviation is thus profitable.

**Step 4:** $\beta(\cdot)$ cannot be strictly increasing and such that $\beta(v) > v$ on some $[v_0, v_0 + \varepsilon_0)$.

Suppose, to the contrary that, for all $\varepsilon \in [0, \varepsilon_0)$, $\beta(\cdot)$ is strictly increasing and $\beta(v_0 + \varepsilon) > v_0 + \varepsilon$. Consider a deviation by bidder $i$ of type $v_0 + \varepsilon$ from $\beta(v_0 + \varepsilon)$ to some $b = \beta(v_0 + \eta) \in [v_0, v_0 + \varepsilon)$. The gain (avoided loss) in the first auction is at least equal to

$$\int_{v_0+\eta}^{v_0+\varepsilon} (\beta(y) - v_0 - \varepsilon)\, dG(y).$$

At worst, this deviation may eliminate bidder $i$'s maximal possible gains in the second auction (namely $\delta \left(v_0 + \varepsilon - v_{-i}^{(1)}\right)$) in the event that his deviation actually induces a change in $R_2$ (that is, when $v_{-i}^{(1)} \in [v_0 + \eta, v_0 + \varepsilon]$). So, the net expected benefit of this deviation is bounded from below by:

$$\begin{aligned} \Delta(\varepsilon) &= \int_{v_0+\eta}^{v_0+\varepsilon} \{[\beta(y) - v_0 - \varepsilon] - \delta\,[v_0 + \varepsilon - y]\}\, dG(y) \\ &= \int_{v_0+\eta}^{v_0+\varepsilon} \{[\beta(y) + \delta y] - (1 + \delta)\,[v_0 + \varepsilon]\}\, dG(y). \end{aligned}$$

The function $\beta(y) + \delta y$ is increasing over $[v_0, v_0 + \varepsilon)$ from $(1 + \delta) v_0$ to $\beta(v_0 + \varepsilon) + \delta\,(v_0 + \varepsilon) > (1 + \delta)\,[v_0 + \varepsilon]$.

So, by choosing the unique $\eta \in (0, \varepsilon)$ such that $\beta(v_0+\eta)+\delta\,(v_0+\eta) = (1+\delta)\,[v_0+\varepsilon]$, we have: $\Delta(\varepsilon) > 0$. Thus, again, the deviation is profitable.

So, these four steps prove that in a neighborhood of $v_0$, $\beta(\cdot)$ is equal to the identity mapping. Hence, since $\beta(\cdot)$ is continuous in $[v_*, v_T)$, it must be the identity mapping over this whole interval. ■

We now show that if a symmetric pure strategy equilibrium exists, then all participating buyers bid up to their true valuations. That is, the characterization of bidding up to one's true valuation extends for all participating types of bidder and there is no discontinuity in the bid function above $v_*$.

**Lemma 2** *: Under the assumptions of Lemma 1, the first-auction equilibrium bid function is continuous on* $[v_*, \overline{v}]$ *and* $\beta(v) = v$ *for all* $v \in [v_*, \overline{v}]$.

**Proof.** In light of Lemma 1, we only need to show that $\beta(\cdot)$ has no points of discontinuity above $v_*$. Suppose, to the contrary, that $v_T < \overline{v}$ is the first point of discontinuity to the right of $v_*$. By Lemma 1 it must be $\beta(v) = v$ for all $v \in [v_*, v_T)$. Thus, given right-continuity, there are two possibilities: either $\beta(v_T) > v_T$, or $\beta(v_T) < v_T$.

**Case 1:** $\beta(v_T) > v_T$.

Take $\varepsilon_T > 0$ such that: $\beta(\cdot)$ is continuous and $\beta(v) > v_T + \varepsilon_T$ for all $v \in [v_T, v_T+\varepsilon_T]$. Let $\varepsilon(\varepsilon_T)$ be the lowest $\varepsilon \in [0, \varepsilon_T]$ such that $\beta(v_T+\varepsilon) = \max_{v\in[v_T, v_T+\varepsilon_T]} \beta(v)$ and let $v(\varepsilon_T)$ be such that $\beta(v(\varepsilon_T)) = \min_{v\in[v_T, v_T+\varepsilon_T]} \beta(v)$. A deviation by type $v_T+\varepsilon(\varepsilon_T)$ from $\beta(v_T+\varepsilon(\varepsilon_T))$ to $v_T+\varepsilon(\varepsilon_T)$ induces a gain in the first auction that is bounded below by

$$\frac{1}{n}\left[\beta(v(\varepsilon_T)) - v_T - \varepsilon(\varepsilon_T)\right]\left[G(v_T+\varepsilon(\varepsilon_T)) - G(v_T)\right].$$

This bound is obtained by multiplying the minimum possible loss from winning with a bid $\beta(v_T+\varepsilon(\varepsilon_T))$ against a type in the interval $[v_T, v_T+\varepsilon_T]$ with a lower bound on the probability of winning when at least one of the other bidders' valuations (but possibly all) is in the interval. Since the deviation can only affect $R_2$ if $v_{-i}^{(1)} \geq v_T$, the loss in the second auction is bounded above by

$$\delta \int_{v_T}^{v_T+\varepsilon(\varepsilon_T)} [v_T + \varepsilon(\varepsilon_T) - v]\, dG(v) < \delta\varepsilon(\varepsilon_T)\,[G(v_T + \varepsilon(\varepsilon_T)) - G(v_T)]\,.$$

The difference between the two bounds yields a lower bound $\Delta(\varepsilon_T)$ on the net gain from the deviation. It is clear that for a sufficiently small $\varepsilon_T$ it is $\Delta(\varepsilon_T) > 0$, and thus the deviation is profitable.

**Case 2:** $\beta(v_T) < v_T$. The proof of this case is similar to the proof of case 1 and is therefore omitted. ■

We have established that if a symmetric, pure strategy, continuation equilibrium exists after a given $R_1$, then types who participate in the first auction bid up to their true valuations. Moreover, in this case Bayesian updating of the seller's beliefs provides, as a corollary, a characterization of the optimal reserve price in the second auction.

- If no bidder participates at $R_1$, then the seller updates his beliefs conditional on $v_i \in [\underline{v}, v_*)$ for all $i$ and optimally sets a reserve price $R_*(v_*)$ given by:

$$R_*(v_*) \equiv R_{\underline{v},v_*} = \begin{cases} J^{-1}_{\underline{v},v_*}(0) & \text{if } J_{\underline{v},v_*}(\underline{v}) < 0 \\ \underline{v} & \text{otherwise.} \end{cases} \tag{3}$$

  Note that, given our assumption, if $R_*(v_*) > \underline{v}$, then:

$$R'_*(v_*) = \frac{f(v_*)}{2f(R_*) + R_* f'(R_*)} > 0. \tag{4}$$

- If the winner is bidder $w$ and $p \geq R_1$ is the ending price of the first auction, then the seller updates his beliefs conditional on $v_w \in [p, \overline{v}]$, on any other $v_i$ being the price at which bidder $i$ dropped from the first auction if $i$ participated and $v_i \in [v, v_*)$ if $i$ did not participate. In other words, the seller gets the information that one bidder has valuation above $p$ while all other bidders have valuations smaller than $p$. Consequently, the optimal reserve price in the second auction $R_2(p)$ corresponds to:

$$R_2(p) \equiv R_{p,\overline{v}} = \sup\{R_0, p\}\,.$$

- If (as will be true) $v_* > R_1$, a winning price $p \in (R_1, v_*)$ is an event off the equilibrium path, for which the seller's beliefs and hence the second auction reserve price are not pinned down by the equilibrium characterization.

Given this continuation strategy by the seller, we are in a position to characterize the set of reserve prices $R_1$ for which there exists a symmetric, pure-strategy bidding equilibrium with active participation in the first auction by a positive measure of bidders' types, who bid up to their valuations. We call such an equilibrium, a semi-separating equilibrium.

**Proposition 3** *: Consider the continuation game for a given $R_1$. A pure strategy, symmetric equilibrium with participation in the first auction by a positive measure of types exists if and only if*

$$R_1 < r_0 \equiv \overline{v}(1-\delta) + \delta \int_{\underline{v}}^{\overline{v}} \sup\{R_0, y\}\, dG(y).$$

*If this condition holds, the equilibrium is unique: bidders participate and bid up to their true valuation in the first auction whenever their valuation is higher than $v_*$ defined by:*

$$\delta \int_{\underline{v}}^{v_*} (v_* - \sup\{y, R_*(v_*)\})\, dG(y) = (v_* - R_1)\, G(v_*), \tag{5}$$

*Below $v_*$, bidders refrain from participating in the first auction.*

**Proof.** Fix $R_1$. From Lemma 2, it remains to be shown that truthful bidding above some $v_*$ is an equilibrium and to characterize $v_*$. Suppose all bidders but bidder $i$ bid accordingly and consider bidder $i$ of type $v_i$.

What are bidder $i$'s expected profits from not participating? If $v_{-i}^{(1)} \geq v_*$, the second auction reserve price will be set at $\sup\left\{v_*, v_{-i}^{(2)}, R_0\right\}$, where $v_{-i}^{(2)}$ is the second rank-order statistics among $(v_1, ...v_{i-1}, v_{i+1}, ...v_n)$. The second-auction sale price conditional on bidder $i$'s winning the second auction will then equal:

$$\sup\left\{v_*, v_{-i}^{(2)}, R_0, v_{-i}^{(1)}\right\} = \sup\left\{R_0, v_{-i}^{(1)}\right\} = R_2(v_{-i}^{(1)}).$$

If $v_{-i}^{(1)} \leq v_*$, however, the reserve price in the second auction will be set equal to $R_*(v_*)$. Overall, bidder $i$'s expected profits from not participating in the first auction are:[9]

$$\delta \int_{\underline{v}}^{v_*} (v_i - \sup\{y, R_*(v_*)\})^+ \, dG(y) + \delta \int_{v_*}^{v_i} (v_i - R_2(y))^+ \, dG(y) \tag{6}$$

where the last integral appears only when $v_i > v_*$.

What if bidder $i$ participates and decides to drop out at price $b$? Suppose first that $b \geq v_*$; that is, $b$ is on the candidate equilibrium path. Either bidder $i$ wins, i.e. $b > v_{-i}^{(1)}$, and the second auction reserve price will be set equal to $\sup\left\{v_*, v_{-i}^{(1)}, R_0\right\} = \sup\{v_*, R_2(v_{-i}^{(1)})\}$, hence independently of $b$. Or he loses, i.e., $b < v_{-i}^{(1)}$, in which case the second auction reserve price is set equal to $\sup\left\{b, v_{-i}^{(2)}, R_0\right\}$. So, in the event where he wins the second auction, that is when $v_i > v_{-i}^{(1)}$, bidder $i$ will pay a price equal to $\sup\left\{v_*, R_2(v_{-i}^{(1)}\right\}$. So, by bidding up to $b \geq v_*$, bidder $i$'s expected profit is:

$$(v_i - R_1)\, G(v_*) + \int_{v_*}^{b} (v_i - y)\, dG(y) + \delta \int_{\underline{v}}^{v_i} (v_i - \sup\{v_*, R_2(y)\})^+ \, dG(y). \tag{7}$$

Suppose then that $b$ is off the equilibrium path: $b \in [R_1, v_*)$. Either bidder $i$ wins, i.e. $v_{-i}^{(1)} < v_*$, and the second-auction reserve price will be set equal to $R_2(v_*)$, independently of $b$. Or, he loses, i.e., $b < v_* \leq v_{-i}^{(1)}$, and then we assume the seller holds beliefs about the bidders' types that leads him to fix $R_2 = r \geq R_0$ (e.g., the seller believes that $i$'s valuation is at least $v_*$). The price bidder $i$ will pay if he wins the second auction is then $\sup\left\{v_{-i}^{(1)}, r\right\} \geq R_2(v_{-i}^{(1)})$ and his expected profit from bidding $b \in [R_1, v_*)$ is:

$$(v_i - R_1)\, G(v_*) + \delta \int_{\underline{v}}^{v_*} (v_i - \sup\{R_2(v_*), y\})^+ \, dG(y) + \delta \int_{v_*}^{v_i} (v_i - \sup\{y, r\})^+ \, dG(y), \tag{8}$$

where the last integral appears only when $v_i > v_*$.

Note that $\sup\{R_2(v_*), y\} = \sup\{v_*, R_2(y)\}$. Hence, for $v_i \geq v_*$, it is immediate from (7) and (8) that bidding $b \geq v_*$ is preferred to bidding $b \in [R_1, v_*)$. Moreover, optimizing w.r.t. to $b$ in (7) yields

[9] We let $(x)^+$ denote $\sup\{x, 0\}$.

$b = v_i$. For $v_i < v_*$, it is also immediate from above that bidding $b = v_*$ is at least as good as bidding any $b \in [R_1, v_*)$, given that $\sup\left\{v_{-i}^{(1)}, r\right\} \geq R_2(v_{-i}^{(1)})$ and preferable to any $b > v_*$. It follows that, if he participates, then the best reply of player $i$ of type $v_i$ is to bid $\sup\{v_i, v_*\}$.

Given the slopes in $v_i$ of the expressions in (6) and (7), and the fact that $[v - R_2(v)]^+ = 0$ for all $v$, there is a unique participation cutoff valuation $v_*$ determined by:

$$\delta \int_{\underline{v}}^{v_*} (v_* - \sup\{y, R_*(v_*)\})\, dG(y) = (v_* - R_1)\, G(v_*), \tag{9}$$

and truthful bidding above this $v_*$ is indeed part of a continuation equilibrium, given $R_1$.

Note that for $v_* = R_1$, the LHS of (9) is positive while the RHS is null. The function given by the difference between the RHS and the LHS in (9) has a total derivative w.r.t. to $v_*$ equal to

$$(1-\delta)G(v_*) + \delta R_*'(v_*)\,[G(v_*) - G(R_*(v_*)] + (v_* - R_1)g(v_*)$$

which is positive for $v_* \geq R_1$. Hence this function is monotone, negative for $v_* = R_1$. Since $R_*(\overline{v}) = R_0$, it follows that a solution exists if and only if:

$$R_1 < \overline{v}(1-\delta) + \delta \int_{\underline{v}}^{\overline{v}} \sup\{R_0, y\}\, dG(y) = r_0.$$

This completes the proof of the proposition. ■

Note that $R_0 < r_0$, so that for $R_1 \leq R_0$, a semi-separating equilibrium always exists. Note also that since $R_*(v_*) < v_*$, if $R_1 > \underline{v}$ then we have $v_* > R_1$; that is, some bidders with valuations higher than the reserve price do not participate in the first auction.

In the first auction, a bidder is concerned that if he turns out to be the highest bidder and to win the first unit, then he might disclose so much information about his valuation that the seller would be in a position to capture all his rent in the second auction, by setting a reserve price equal to his valuation. An ascending-price auction format, however, is ample enough protection against the seller's opportunism, as it only provides a lower bound on the winner's valuation. Bidding truthfully in an ascending auction amounts to perfectly revealing one's own valuation only when one does not win; but if a participating bidder does not win the first auction, it is because there is another bidder with a higher valuation. Thus, when losing in the first auction the bidder knows that he will also lose in the second auction and does not care about revealing his valuation. In other words, there is no point for a bidder to get out of the first auction when another bidder is still in and the price has not yet reached his true valuation.

However, if no other bidder participates in the first auction, it may make sense for a bidder with a value above the first-auction reserve to not participate either, in order to induce the seller to decrease the second-auction reserve price. By not participating when nobody else does, a bidder induces the seller to hold more pessimistic beliefs; namely, to believe that all bidders have valuation below some threshold $v_*$ instead of being identified as the only bidder with valuation higher than $v_*$. When $v$ is close to $R_1$, the bidder has not much to expect from the first auction; by not participating he can influence the second-auction reserve price and raise his expected profits. This explain why the participation threshold $v_*$ is strictly higher than the first-auction reserve $R_1$. The threshold value $v_*$ is precisely the type who is indifferent between $v_* - R_1$ with probability $G(v_*)$ today and the expected profit that he would make tomorrow by not participating today.

Note that although truthful bidding is an equilibrium behavior for the participating bidders in the first auction, it is by no means a dominant strategy. As we shall see in Section 5, in sealed-bid auctions, as in the standard contracting literature without commitment, truthful bidding is impossible, since monotone bidding functions would reveal the winning bidder's true valuation through his first-auction bid, and hence would wipe out his informational rent in the second auction.

## 3.2 Full Pooling with No Participation in The First Auction

We now investigate whether there exist continuation equilibria with no bidders participating in the first auction. In this case, it is clear that in equilibrium the seller does not update his beliefs and will optimally fix a reserve price equal to $R_0$ in the second auction.

**Proposition 4** *: Consider the continuation game with a given $R_1$. There exists a full pooling equilibrium where bidders refrain from participating if and only if $R_1 \geq r_0$ and, in this case, there is no other pure-strategy symmetric continuation equilibrium.*

**Proof.** Given Proposition 3, it remains to show that non-participation is an equilibrium strategy when $R_1 \geq r_0$. If nobody else participates, bidder $i$ with valuation $v_i$ then simply expects the following equilibrium payoff:

$$\delta \int_{\underline{v}}^{v_i} (v_i - R_2(y))^+ \, dG(y)$$

from non-participating. By deviating, i.e. by participating, he could win the first auction for a price of $R_1$. Since participation is off the candidate equilibrium path, assume the seller's beliefs after observing participation are concentrated on the highest type $\overline{v}$, which results in a take-it-or-leave-it offer at price $\overline{v}$ in the second auction. (Note that these beliefs provide the least incentives for participation and thus are the most favorable for the existence of a pooling equilibrium with no bidders participating.) The bidder then earns no profits from the second auction. The equilibrium condition is then: for all $v_i$,

$$\delta \int_{\underline{v}}^{v_i} (v_i - R_2(y))^+ \, dG(y) \geq v_i - R_1.$$

Given that the LHS increases less steeply with $v_i$ than the RHS, a necessary and sufficient condition is therefore that the above condition holds for $\overline{v}$:

$$R_1 \geq \overline{v}(1-\delta) + \delta \int_{\underline{v}}^{\overline{v}} R(y) dG(y) = r_0.$$

This completes the proof. ■

Therefore, for a given value of $R_1$, there exists a unique pure-strategy, symmetric equilibrium. Unsurprisingly, if the reserve price in the first auction is sufficiently large, namely above $r_0$, then bidders do not participate. Their prospects from first-auction participation are too low compared to the potential benefits of winning the second unit in a second auction with a low reserve price. If, on the other hand, the reserve in the first auction is sufficiently low, below $r_0$, then equilibrium is semi-separating (monotone increasing for high types), with bidders having high valuations (i.e., values $v_i \geq v_*$) participating in the first auction and bidding up to their valuations.

# 4 Equilibrium and its Properties

The previous section characterized the (pure-strategy, symmetric) continuation equilibria for a given reserve price $R_1$ chosen by the seller in the first auction. It is now a straightforward task to characterize the global equilibrium, namely the equilibrium reserve price $R_1$ endogenously chosen by the seller, and the ensuing continuation equilibrium. We will show that the seller always induces participation by a positive measure of bidders' types; formally, $v_* < \overline{v}$. We also show that, in the first auction, there is less equilibrium participation than in the optimal one-shot auction; formally, $v_* > R_0$. On the contrary, the probability that at least one bidder participates in the second auction is higher than in the optimal auction; formally, $R_*(v_*) < R_0$.

Recall first that (5) provides a one-to-one mapping from $R_1$ to the ensuing equilibrium participation threshold $v_*$. It is indeed more convenient to think that the seller sets directly the participation threshold $v_*$. Rearranging (5) we then obtain that the reserve price in the first auction equal to:

$$R_1 = v_* - \delta \int_{R_*(v_*)}^{v_*} \frac{G(v)}{G(v_*)} dv. \tag{10}$$

Recall also that (3) determines $R_*(v_*)$, that is the second-auction reserve price whenever no participation took place in a first auction characterized by the participation threshold $v_*$, while $R_2(p) = \sup\{p, R_0\}$ is the equilibrium reserve price in the second auction after the first auction concluded at a final price $p \geq v_*$. Characterizing the global equilibrium amounts then to determining the equilibrium participation threshold $v_*$. To simplify the notation, in the rest of this section we will omit the argument of $R_*(v_*)$ and simply write $R_*$.

**Proposition 5** *: There exists generically a unique perfect Bayesian equilibrium (with pure-strategy, symmetric, continuation equilibria). It corresponds to a participation threshold $v_*$ that solves:*

$$\delta G(R_*) R_*' [1 - F(R_*) - R_* f(R_*)] + G(v_*) [1 - F(v_*) - v_* f(v_*)] = 0. \tag{11}$$

*Furthermore, if $f(\bar{v}) > 0$ and $R_0 > \underline{v}$ the following strict inequalities hold:*

$$R_* < R_0 < v_* < \bar{v},$$

*while if $R_0 = \underline{v}$, then $v_* = R_* = R_0 = \underline{v}$.*

**Proof.** Using the fact that the joint density of $(v^{(1)}, v^{(2)})$ is given by $nf(v^{(1)})g(v^{(2)})\mathbf{1}_{\{v^{(2)} \leq v^{(1)}\}}$, we can write the seller's expected profits as follows (the first integral relates to $v^{(2)}$, the second to $v^{(1)}$):

$$\begin{aligned}
\pi(R_*, v_*) &= \delta n \int_{\underline{v}}^{v_*} \int_{\sup\{R_*, y\}}^{v_*} \sup\{R_*, y\} f(x) g(y) dx dy \\
&\quad + nR_1 \int_{\underline{v}}^{v_*} \int_{v_*}^{\bar{v}} f(x) g(y) dx dy + \delta n R_2(v_*) \int_{\underline{v}}^{v_*} \int_{R_2(v_*)}^{\bar{v}} f(x) g(y) dx dy \\
&\quad + n \int_{v_*}^{\bar{v}} \int_{y}^{\bar{v}} y f(x) g(y) dx dy + \delta n \int_{v_*}^{\bar{v}} \int_{R(y)}^{\bar{v}} R_2(y) f(x) g(y) dx dy.
\end{aligned}$$

Integrating over $x$ and replacing $R_1$ using equation (10), one obtains:

$$\begin{aligned}
\pi(R_*, v_*) &= \delta n \int_{\underline{v}}^{v_*} \sup\{R_*, y\} [F(v_*) - F(\sup\{R_*, y\})] g(y) dy \\
&\quad + n [1 - F(v_*)] \left[ v_* G(v_*) - \delta \int_{R_*}^{v_*} G(y) dy \right] + \delta n R_2(v_*) [1 - F(R_2(v_*))] G(v_*) \\
&\quad + n \int_{v_*}^{\bar{v}} y [1 - F(y)] g(y) dy + \delta n \int_{v_*}^{\bar{v}} R_2(y) [1 - F(R_2(y))] g(y) dy.
\end{aligned}$$

Recalling that $J(v) = v - \frac{1-F(v)}{f(v)}$, it follows that:

$$\frac{1}{n}\frac{\partial \pi}{\partial v_*} = -f(v_*)J(v_*)G(v_*)\left[1-\delta+\delta \mathbf{1}_{\{v_* \geq R_0\}}\right],$$

$$\frac{1}{n}\frac{\partial \pi}{\partial R_*} = -\delta G(R_*)f(R_*)J(R_*).$$

Suppose first that $v_* < R_0$, then given our monotone likelihood property assumption: $J(R_*) < J(v_*) < J(R_0) = 0$. Since $R_*' > 0$, it follows that $\frac{d\pi}{dv_*} > 0$. It is not optimal for the seller to choose $R_1$ so as to induce $v_* < R_0$.

Suppose then that $v_* \geq R_0 > \underline{v}$. For $v_* = R_0$, $J(R_*) < J(v_*) = J(R_0) = 0$ and so,

$$\frac{1}{n}\frac{d\pi}{dv_*}\mid_{v_*=R_0} = -\delta f(R_*)J(R_*)G(R_*)R_*'(R_0) > 0.$$

This implies that in equilibrium $v_* > R_0$. On the other hand, for $v_* = \bar{v}$, $R_* = R_0$ and $J(R_*) = 0$ while $J(v_*) > 0$. It follows that

$$\frac{1}{n}\frac{d\pi}{dv_*}\mid_{v_*=\bar{v}} = -f(\bar{v})\bar{v} < 0.$$

Hence, in equilibrium, $v_* < \bar{v}$ and there is participation in the first auction. The equilibrium participation threshold $v_*$ is given by the maximum condition $v_* \in \arg\max_x \pi(R_*(x), x)$, hence the generic uniqueness. It follows also that the FOC:

$$-\delta f(R_*)J(R_*)G(R_*)R_*' - f(v_*)J(v_*)G(v_*) = 0$$

holds, where $R_*(v_*)$ depends on $v_*$. This implies that if $R_0 > \underline{v}$, then $R_*(v_*) < R_0 < v_* < \bar{v}$. If $R_0 = \underline{v}$, then, clearly, $v_* = R_* = R_0 = \underline{v}$. This completes the proof. ■

Proposition 5 asserts that in equilibrium the seller strategically chooses the first-auction reserve price so as to induce a positive measure of bidders' types to participate, but less than in a one-shot auction; that is, less than if he were able to commit not to use in the second-auction the information revealed by the different bids in the first auction. Non-commitment therefore induces a reduction in participation in the first auction. Although truthful bidding is still an equilibrium strategy for participating bidders, strategic retention of information takes the form of intermediate-valuation bidders refraining from competing in the first auction to avoid revealing information on their types, compared to the situation where this information would not be used.

Although participation is the relevant measure of the first-auction performance, note that our results do not imply that the actual reserve price chosen in the first auction is larger than when the seller can perfectly commit to the sequence of reserve prices. Indeed, it is possible that $R_1 < R_0$ and still $R_0 < v_*$. A simple examination of (10) allows us to assert that the difference $(v_* - R_1)$ decreases when $n$ increases, but no result with respect to the comparison with $R_0$ is available.[10]

Comparative static results with respect to $\delta$ are also immediate by inspection of (10) and (11). When $\delta$ vanishes, the equilibrium converges to the one-shot auction equilibrium. When $\delta$ increases, $v_*$ increases above $R_0$ but stays bounded away from $\bar{v}$, while $R_*$ increases and stays bounded away from $R_0$. Strategic manipulation through non-participation in the first auction becomes more costly.

It is also clear that the seller cannot make a higher expected revenue in the repeated ascending auction without commitment than in the optimal dynamic auction, with commitment on reserve prices. We show, indeed, that the seller makes strictly less; that is, there is a positive cost for the seller to his lack of commitment power over reserve prices.

**Proposition 6** : *The seller's expected revenue in the repeated ascending auction is strictly less than in the optimal sequential auction under full commitment.*

**Proof.** >From the proof of Proposition 5, $\frac{\partial \pi}{\partial R_*}(R,v) > 0 > \frac{\partial \pi}{\partial v_*}(R,v)$ for all $(R,v)$ such that $R_* \leq R < R_0 < v \leq v_*$. Therefore, $\pi(R_*, v_*) < \pi(R_0, R_0)$. Since $\pi(R_0, R_0)$ is precisely the seller's expected revenue in the optimal sequential auction under full-commitment, the proposition follows. ■

[10] It is also immediate and no surprise that: $\lim_{n \longrightarrow \infty} v_* = \lim_{n \longrightarrow \infty} R_1 = R_0$.

## 5 Sequential Sealed-Bid Auctions

We now focus on a situation with two sequential, second-price auctions. The critical difference with the case of sequential ascending auctions is that the seller observes all $n$ bids, as opposed to just observing the $n-1$ losing bids. This has important implications for the reserve price that the seller sets in the second auction and, consequently, on the equilibrium in the first auction. We prove that for any value of $R_1$, there cannot exist any symmetric, pure strategy, bidding equilibrium in the first auction with weakly increasing bidding functions, conditional on participation, apart from a full pooling equilibrium among participating types. In particular, full separation among participating types is impossible. In contrast with the previous sections, these results are perfectly in line with the literature on the ratchet effect, in particular with Laffont-Tirole [1988].[11]

As for sequential ascending auctions, it is a weakly dominant strategy for bidders in the second auction to participate and bid their true valuation whenever it exceeds $R_2$. Let $R_1$ be the first auction reserve price. We only consider symmetric, pure strategy equilibria of this continuation game, such that the first auction equilibrium bidding function $\beta(\cdot)$ is weakly increasing whenever relevant, that is whenever the bid exceeds $R_1$.[12] We are able to prove the following result.

**Proposition 7** *: There exists $r_1 < r_0$ such that if $r_1 \leq R_1 \leq r_0$, there exists a unique symmetric, pure strategy, weakly increasing equilibrium in the first auction; all types $v$ above a threshold $v_*(R_1)$ participate and bid $\beta(v) = v_*(R_1)$, while types below $v_*(R_1)$ do not participate in the first auction. If $R_1 < r_1$, no symmetric, pure-strategy, weakly increasing equilibrium exists.*

**Proof.** Suppose first there exists a symmetric, pure-strategy, weakly increasing equilibrium, with

[11] As a consequence, the proofs in this section do not provide many details especially as they follow similar steps as in the previous sections, with similar notation.

[12] It is easy to eliminate the possibility of weakly-decreasing equilibrium bidding functions $\beta(\cdot)$, but we have not been able to eliminate the possibility of non-monotonic equilibria. Given our purpose of stressing the difference with the case of sequential ascending auctions, considering only weakly increasing equilibria seems a mild restriction.

bidding function $\beta(\cdot)$. Let $v_*$ be the lowest type participating in the first auction and $v_T$ the lowest discontinuity point of $\beta(\cdot)$ above $v_*$. As in Lemma 1, $\beta(v_*) = v_*$ (which imposes $v_* \geq R_1$).

**Step 1:** There must exist $v_M \in [v_*, v_T)$ such that $\beta(v) = v$ for $v \in [v_*, v_M]$ and $\beta(v) = v_M$ for $v \in [v_M, v_T)$.

Fix any $v_0 \in [v_*, v_T)$ and suppose that $\beta(\cdot)$ is strictly increasing on some $[v_0, v_0 + \varepsilon_0)$. A bid $\beta(v_0 + \varepsilon)$ for any $\varepsilon \in [0, \varepsilon_0)$ reveals the type $v_0 + \varepsilon$, so that the second auction reserve price after observing a (non-necessarily winning) bid $\beta(v_0 + \varepsilon)$ must be larger or equal to $v_0 + \varepsilon$. If, for a positive measure of $\varepsilon$, $\beta(v_0 + \varepsilon) \neq v_0 + \varepsilon$, a deviation to bid $v_0 + \varepsilon$ entails no loss in the second auction and yields a positive expected gain in the first auction, because with positive probability the opponents' highest bid lies in the interval with endpoints $\beta(v_0 + \varepsilon)$ and $v_0 + \varepsilon$. So, $\beta(v) = v$ on $[v_0, v_0 + \varepsilon_0)$ and continuity on $[v_*, v_T)$ yields the result.

**Step 2:** $\beta(\cdot)$ must be continuous on $[v_*, \bar{v}]$.

Suppose there is a jump up at $v_T$. Suppose moreover that $\beta(\cdot)$ is strictly increasing in $[v_*, v_T)$. Then, for $\varepsilon$ small enough, a bid $\beta(v_T + \varepsilon) > v_T$ implies $R_2 \geq v_T + \varepsilon$. A deviation to a bid $v_T + \varepsilon$ by type $v_T + \varepsilon$ entails no loss in the second auction and generates a gain with positive probability in the first auction. Thus, if there is a jump at $v_T$, then $\beta(v) = v$ for all $v \in [v_*, v_M]$ and $\beta(v) = v_M$ for all $v \in [v_M, v_T)$, with $v_M < v_T$. A similar argument implies that the jump must be to $v_T$; that is, $\beta(v_T) = v_T$, and type $v_T$ must be indifferent between bidding $v_M$ and bidding $v_T$.

Consider now types $v_T - \varepsilon$, for $\varepsilon$ small enough. Recall that after observing winning bid(s) equal to $v_M$, the seller optimally sets $R_2 = R_{v_M,v_T}$. Bidding $v_M$ therefore yields:

$$\begin{aligned}\pi^E &= (v_T - \varepsilon - R_1)G(v_*) + \int_{v_*}^{v_M} (v_T - \varepsilon - y)dG(y) + (v_T - \varepsilon - v_M)T(v_M, v_T) \\ &\quad + \delta(v_T - \varepsilon - R_{v_M,v_T})G(R_{v_M,v_T}) + \delta \int_{R_{v_M,v_T}}^{v_T-\varepsilon} (v_T - \varepsilon - y)dG(y),\end{aligned}$$

where $T(v_M, v_T)$ is the probability that a bid $v_M$ wins when at least one of the other players also bids

$v_M$:

$$\begin{aligned} T(v_M, v_T) &= \sum_{k=1}^{n-1} \frac{1}{k+1} \binom{n-1}{k} [F(v_T) - F(v_M)]^k [F(v_M)]^{n-1-k} \\ &= \frac{[F(v_T)]^n - [F(v_M)]^n}{n(F(v_T) - F(v_M))} - G(v_M). \end{aligned}$$

A deviation by types $v_T - \varepsilon$ to the bid $v_T - \varepsilon$ yields at least (with zero gains in the second auction):

$$\pi^D \geq (v_T - \varepsilon - R_1)G(v_*) + \int_{v_*}^{v_M} (v_T - \varepsilon - y)dG(y) + (v_T - \varepsilon - v_M)D(v_M, v_T), \tag{12}$$

where $D(v_M, v_T)$ is the probability that the highest bid of the other players is $v_M$:

$$D(v_M, v_T) = \sum_{k=1}^{n-1} \binom{n-1}{k} [F(v_T) - F(v_M)]^k [F(v_M)]^{n-1-k} = G(v_T) - G(v_M).$$

The gain from the deviation $\pi^D - \pi^E$ is at least equal to:

$$\Delta(\varepsilon) \equiv (v_T - \varepsilon - v_M) [D(v_M, v_T) - T(v_M, v_T)] - \delta(v_T - \varepsilon - v_M)G(v_M) - \delta \int_{v_M}^{v_T - \varepsilon} (v_T - \varepsilon - y)dG(y).$$

Differentiating with respect to $\varepsilon$ we have

$$\Delta'(\varepsilon = 0) = -[D(v_M, v_T) - T(v_M, v_T)] + \delta G(v_T).$$

For type $v_T$, $\pi^D$ satisfies (12) as an equality with $\varepsilon = 0$ and since $v_T$ must be indifferent between bidding $v_T$ and bidding $v_M$, we have $\Delta(0) = 0$, and so:

$$D(v_M, v_T) - T(v_M, v_T) - \delta G(v_M) = \delta \int_{v_M}^{v_T} \frac{v_T - y}{v_T - v_M} dG(y) < \delta G(v_T) - \delta G(v_M).$$

It follows that:

$$0 > D(v_M, v_T) - T(v_M, v_T) - \delta G(v_T) = -\Delta'(\varepsilon = 0).$$

Hence $\Delta'(\varepsilon = 0) > 0$ and $\Delta(\varepsilon) > 0$ for $\varepsilon$ sufficiently small; the deviation is profitable for type $v_T - \varepsilon$. Consequently, there cannot be a jump up in the bid function.

**Step 3:** There cannot exist $v_M > v_*$ such that $\beta(v) = v$ on $[v_*, v_M]$ and $\beta(v) = v_M$ on $[v_M, \bar{v}]$.

Suppose $\beta(v) = v$ on $[v_*, v_M]$. After observing a winning bid $v \in [v_*, v_M]$, $R_2 = v$. Consider $v_0 \in [v_*, v_M)$. Bidding $\beta(v) = v$ yields:

$$\pi^E = (v_0 - R_1)G(v_*) + \int_{v_*}^{v_0} (v_0 - y)dG(y).$$

Deviating on a bid $v_0 - \varepsilon \geq v_*$ yields at :

$$\begin{aligned}\pi^D &= (v_0 - R_1)G(v_*) + \int_{v_*}^{v_0-\varepsilon} (v_0 - y)dG(y) \\ &\quad +\delta\varepsilon G(v_0 - \varepsilon) + \delta \int_{v_0-\varepsilon}^{v_0} (v_0 - y)dG(y).\end{aligned}$$

The gain from the deviation is

$$\Delta(\varepsilon) = \delta\varepsilon G(v_0 - \varepsilon) - (1 - \delta) \int_{v_0-\varepsilon}^{v_0} (v_0 - y)dG(y).$$

Note that $\Delta(0) = 0$, $\Delta'(\varepsilon = 0) = \delta G(v_0) > 0$. Hence $\Delta(\varepsilon) > 0$ for $\varepsilon$ sufficiently small and the deviation is profitable for type $v_0$.

Thus, the only possible symmetric pure strategy equilibrium is $\beta(v) = v_*$ for all $v \in [v_*, \overline{v}]$ and no participation of types below $v_*$.

**Step 4:** Determination of $v_*$.

Type $v_*$'s payoff from not participating is

$$\delta \int_{\underline{v}}^{v_*} (v_* - \sup\{y, R_*(v_*)\})\, dG(y),$$

while the payoff from participating is

$$(v_* - R_1)\,[G(v_*) + T(v_*, \overline{v})] = (v_* - R_1)\,\frac{[1 - [F(v_*)]^n]}{n\,[1 - F(v_*)]}.$$

Indifference of type $v_*$ between participating and not participating in the first auction gives:

$$\delta \int_{\underline{v}}^{v_*} (v_* - \sup\{y, R_*\})\, dG(y) = (v_* - R_1)\,\frac{[1 - [F(v_*)]^n]}{n\,[1 - F(v_*)]}. \tag{13}$$

Following similar steps as in the discussion of (9) in the proof of Proposition 3, and noting that $\frac{[1-[F(v_*)]^n]}{n[1-F(v_*)]}$ is larger than $G(v_*) = [F(v_*)]^{n-1}$ and is equal to 1 when $v_* = \bar{v}$, it follows that (13) defines a unique increasing solution $v_*(R_1)$ within $(R_1, \bar{v})$ if and only $R_1 < r_0$, and has no solution when $R_1 > r_0$ (in which case the LHS is larger than the RHS for all $v_*$).

**Step 5:** Full pooling above $v_*$ is an equilibrium.

Suppose $R_1$ is such that (13) determines $v_*(R_1) < R_0$. After observing a bid $v_*$ in the first auction, the seller sets a second period reserve equal to $R_{v_*,\bar{v}} = R_0$. Then, type $v$ such that $v_* < v < R_0$ is better off deviating to the bid $v$ rather than bidding $v_*$. Thus, full pooling above $v_*$ can only be an equilibrium if $R_1$ is such that $v_*(R_1) \geq R_0$. This condition implies that $R_1$ be large enough, with a lower bound smaller than $r_0$ since for $R_1 = r_0$, $v_* = \bar{v} > R_0$.

Suppose then that $R_1$ is such that $v_*(R_1) \geq R_0$; this implies $R_2 = v_*$ if at least one bidder bids $v_*$ in the first auction. In the pooling equilibrium, the equilibrium payoff of type $v$, with $v \geq v_*$, is:

$$(v - R_1)G(v_*) + (v - v_*)T(v_*, \overline{v}) + \delta(v - v_*)G(v_*) + \delta \int_{v_*}^{v} (v - y)dG(y).$$

The payoff from a deviation to the bid $b > v_*$, when the seller's beliefs induce $R_2 = \overline{v}$ (so as to make any deviation as unprofitable as possible) is:

$$(v - R_1)G(v_*) + (v - v_*)D(v_*, \overline{v}).$$

The gain from the deviation is

$$\Delta(v; v_*) = (v - v_*) [D(v_*, \overline{v}) - T(v_*, \overline{v})] - \delta(v - v_*)G(v_*) - \delta \int_{v_*}^{v} (v - y)dG(y).$$

Differentiating with respect to $v$ we have

$$\Delta_1(v; v_*) = D(v_*, \overline{v}) - T(v_*, \overline{v}) - \delta G(v),$$

which is a decreasing function of both $v$ and $v_*$. Let $v_Z$ be the unique solution of $\Delta_1(v_Z; v_Z) = 0$. Note that necessarily, $\underline{v} < v_Z < \bar{v}$. Then, observing that $\Delta(v_*; v_*) = 0$, $\Delta_1(v; v_*) \leq 0$ for all $v \geq v_* \geq v_Z$, while there exist $v$ and $v_*$ such that $\Delta_1(v; v_*) > 0$ if $v_* < v_Z$. Thus, $\beta(v) = v_*$ for all $v \in [v_*, \overline{v}]$ is an equilibrium if and only if $v_* \geq v_Z$ (provided $v_* \geq R_0$). This last condition implies that $R_1$ be large enough, with a lower bound smaller than $r_0$, since $v_Z < \bar{v}$. Setting $r_1$ as the supremum of the two lower bounds found in step 5 completes the proof. ■

Observing all the bids, in particular the winning bid, therefore implies drastically different dynamic considerations in terms of information disclosure imbedded in the first auction bidding behavior. The only (weakly increasing) bidding equilibrium with participation involves only the coarsest revelation of information, i.e. pooling among all participating type.

When $R_1$ is small, namely $R_1 < r_1$, no symmetric, pure-strategy (weakly increasing) equilibrium exists.[13] When $R_1$ is large, however, as in the case of ascending auctions, all bidders prefer to abstain from participating in the first auction and strategically wait for the second auction. The next proposition is the counterpart of Proposition 4; the proof is analogous and is omitted.

**Proposition 8** *: If $R_1 > r_0$, there exists only one symmetric, pure-strategy equilibrium; no bidder participates in the first period auction.*

# 6 Conclusions

We have shown that the equilibrium of the sequential ascending auction is much simpler than the equilibrium of the sequential, sealed-bid, second-price auction. In particular, it only exhibits pooling in the form of strategic non-participation in the first auction by low valuation bidders. Thus, we have provided rigorous content to the intuition by Ausubel and Milgrom (2001) that, from the point of view of protecting bidders' against future exploitation, the ascending auction is theoretically superior. This also helps to explain why the ascending auction is so popular in practice.

Contrary to the Vickrey auction, whose practical use has been rare (see Rothkopft et al. (1990)),

[13] Constructing an equilibrium appears to be a difficult task in this case. Our attempts have not succeded.

the first price auction is also often used in practice. While we have not analyzed sequential, first-price auctions, it seems clear that the ratchet effect would be at least as serious as in second-price auctions. In addition to having an incentive to conceal information from the seller, bidders would also want to conceal information from each other, in order to affect bidding in their favor in the second auction. However, as argued first by Robinson (1985) the oral ascending auction is more susceptible than the first-price auction to bidders' collusion (see Brusco and Lopomo (2002) for a recent application). Thus, while our analysis supports the view that the ascending auction is superior to the other auction forms if the bidders' concerns of future exploitation are a dominant factor, the first-price auction is superior when the seller's fear of bidders' collusion is the dominant factor.

Two important limitations of our model are that it only considers a sequence of two auctions and that bidders' valuations are perfectly correlated across time. In our view each of these limitations is less problematic in view of the other. Under perfect correlation a bidder's reluctance to reveal his valuation is most severe. Thus, given our focus on assessing the extent to which the ascending-price auction minimizes pooling of bids and complex equilibrium behavior, perfect correlation seems the appropriate assumption. Once one has assumed perfect correlation, it is natural to look at a sequence of two auctions, for the following reason. With perfect correlation of valuations, any sequence of auctions in which the highest bidder becomes known degenerates into a sequence of fixed-price offers by a single seller to a single buyer. Thus, before analyzing a model with an arbitrary, finite sequence of auctions we would need to solve the single-seller, single-buyer model. Unfortunately, as pointed out by Laffont and Tirole (1993, p. 407), "little is known about the equilibrium path for arbitrary horizons" of the single-seller, single-buyer, fixed-price contracting model without commitment. In our view, a more promising research direction is both to relax perfect correlation of a bidder's valuation across time and to allow for an arbitrary finite number of auctions. We plan to pursue this line of research in the future.

## References


[1] Ashenfelter, O., 1989: "How Auctions Work for Wine and Art," *Journal of Economic Perspectives*, 3, 23-36.

[2] Ausubel L. and P. Milgrom, 2001: "Ascending Auctions with Package Bidding", mimeo.

[3] Black, J., and D. de Meza, 1992: "Systematic Price Divergences Between Successive Auctions are no Anomaly," *Journal of Economics and Management Strategy*, 1, 607-28.

[4] Brusco, S., and P. Lopomo, 2002: "Collusion via Signaling in Ascending Auctions with Multiple Objects and Complementarities," *Review of Economic Studies*, forthcoming.

[5] Burguet, R., and J., Sakovics, 1996: "Reserve Prices without Commitment," *Games and Economic Behavior*, 15, 149-64.

[6] Cassady, R. Jr., 1967: *Auctions and Auctioneering*, University of California Press, Berkeley and Los Angeles.

[7] Donald, S., Paarsch, H., and J. Robert, 1997: "Identification, Estimation and Testing in Empirical Models of Sequential Ascending-Price Auctions with Multi-Demand: An Application to Siberian Timber-Export Permits," mimeo.

[8] Donald, S., Paarsch, H., and J. Robert, 2001: "Identification and Estimation in Empirical Models of Sequential, Oral, Ascending-Price Auctions," mimeo, University of Iowa.

[9] von der Fehr, N.H., and N.H. Morch, 1994: "Predatory Bidding in Sequential Auctions," *Oxford Economic Papers*, 46, 345-56.

[10] Gale, I., and D.B. Hausch, 1994: "Bottom-Fishing and Declining Prices in Sequential Auctions," *Games and Economic Behavior*, 7, 318-31.

[11] Gale, I., Hausch, D.B., and M. Stegeman, 2000: "Sequential Procurement with Subcontracting," *International Economic Review*, 41, 989-1020.

[12] Gale, I., and M. Stegeman, 2001: "Sequential Auctions of Endogenously Valued Objects," *Games and Economic Behavior*, 36, 74-103.

[13] Gandal, N., 1997: "Sequential Auctions of Independent Objects: Israeli Cable Television Licences," *Journal of Industrial Economics*, 45(3), 227-44.

[14] Haile, P.A., 2000: "Partial Pooling at the Reserve Price in Auctions with Resale Opportunities, *Games and Economic Behavior*, 22, 231-48.

[15] Hausch, D., 1986: "Multi-Object Auctions: Sequential vs. Simultaneous Sales," *Management Science*, 32, 4599-610.

[16] Jeitschko, T.D., 1999: "Equilibrium Price Paths in Sequential Auctions with Stochastic Supply," *Economics Letters*, 64, 67-72.

[17] Katzman, B., 1999: "A Two Stage Sequential Auction with Multi-Unit Demands," *Journal of Economic Theory*, 86, 77-99.

[18] Klemperer, P.D., 1999: "Auction Theory: A Guide to the Literature," *Journal of Economic Surveys*, 13, 227-86.

[19] Laffont, J.J., Ossard, H., and Q. Vuong, 1995: "Econometrics of First-Price Auction," *Econometrica*, 63, 953-80.

[20] Laffont, J.J., and J. Tirole, 1988: "The Dynamics of Incentive Contracts," Econometrica, 56, 1153-1176.

[21] Laffont J.J., and J. Tirole, 1993: *A Theory of Incentives in Procurement and Regulation*, MIT Press, Cambridge, USA.

[22] Luton, R., and R.P. McAfee, 1986: "Sequential Procurement Auctions," *Journal of Public Economics*, 31, 181-95.

[23] McAfee, R.P., and D.Vincent, 1993: "The Declining Price Anomaly," *Journal of Economic Theory, 60, 191-211.*

[24] McAfee, R.P., and D.Vincent, 1997: "Sequentially Optimal Auctions," *Games and Economic Behavior, 18, 246-76.*

[25] Milgrom, P.R., 2000: "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, 108, 245-272.

[26] Milgrom, P.R., 2001: "*Putting Auction Theory to Work*," mimeo.

[27] Montmarquette, C., and J. Robert, 1999: "Sequential Auctions with Multi-Unit Demand: Theory, Experiments and Simulation," mimeo CIRANO Scientific Series #99-46, Montreal.

[28] Myerson, R.B., 1981: "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

[29] Ortega Reichert, A., 1968: "A Sequential Game with Information Flow," Chapter VIII of Ph.D. Thesis, Stanford University, reprinted in *The Economic Theory of Auctions*, 2000, P.Klemperer ed., Volume 1, Edward Edgar Pub., Cambridge, UK.

[30] Robinson, M.S., 1985: "Collusion and The Choice of Auction," Rand Journal of Economics, 14, 141-45.

[31] Rothkopf, M.H., Teisberg, T.J., and E.P. Kahn, 1990: "Why Are Vickrey Auctions Rare?," *Journal of Political Economy*, 98, 94-109.

[32] Weber, R., 1983: "Multi-Object Auctions," in *Auctions, Bidding and Contracting*, R.Engelbrecht-Wiggans, M.Shubik and R.Stark eds., New York University Press, New York.