

Auctions and shareholdings*

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Abstract

We study how shareholdings affect revenue and efficiency in standard auction formats with independent private values. Two types of shareholdings are analyzed: Vertical (resp: horizontal) toeholds cover situations in which bidders own a fraction of the seller's profit (resp: a share of their competitor's profit). In general, the expected revenue is an increasing function of vertical toeholds and a decreasing function of horizontal toeholds. However, with both types of toeholds, the auction formats are not equivalent. The second-price auction is more sensitive than the first-price auction to the revenue variations upwards or downwards due to these toeholds.

Keywords : auctions, private values, toeholds, revenue comparison. **JEL classification:** D44, D82, G32, G34.

1. Introduction

We study how shareholdings and cross-shareholdings affect firms behaviors in an auction context. But first, we illustrate why shareholdings can matter in an auction framework through two cases.

The Global One case. Global One was a joint-venture created in 1996 by Deutsche Telekom (40%), France Télécom (40%) and Sprint (20%). In 1999, in

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order to satisfy competition rules, Sprint was forced to sell its shares of Global One. Because of a former agreement, the two remaining ex-partners were the only possible buyers for Sprint's shares. Neither of the two European companies would have accepted to stay with 40% of Global One while the other would have owned 60%. The former partners agreed that they had to choose a selling process such that, at the end, only one of the European operators would hold 100% of Global One. The three firms considered using an auction process.¹ However, the situation was slightly different from the standard auction case. In that case, the bidders, France Télécom and Deutsche Telekom, were wearing two hats: potential buyers of the remaining shares and potential seller of their own shares. Our objective is to study the impact of this extra seller hat.

We have evoked a situation in which bidders own, ex ante, a fraction of the good for sale. This is not the only situation in which shareholdings affect strategic incentives in an auction framework. We also observe situations in which bidders own a fraction of the capital of other bidders. This can also be a major issue. We illustrate that point through a second case.

RVI and Volvo Trucks were two of the main European truck manufacturers. Between 1991 and 1994, they had symmetric crossholdings of 45%. Meanwhile, in some European countries, their joint market share exceeded 50%. Nevertheless, during that period, they still regularly competed with each other in tenders organized by the major haulage companies of these countries. Once again, one can wonder how these toeholds influenced the strategic behaviors of RVI and Volvo Trucks in the tender stages. As a matter of fact, each of the truck manufacturer got back a fraction of the profit of the other. Then, if RVI lost a market, he preferred Volvo Trucks to obtain it and to make the highest possible profit with this tender.

In both cases, Global One and RVI/Volvo Trucks, because of these toeholds, bidders, conditional on losing the auction, care about the price paid by the winner. We intend to understand how it affected bidders' strategies. Do shareholdings affect efficiency or expected revenue? Do they have the same impact on the different auction formats? In concrete terms, which of the standard auction formats is preferable, in terms of efficiency and revenue in that case? This paper addresses all these questions.

We consider two types of toeholds. When a bidder owns shares of the seller

¹The author spoke with the Finance director of France Télécom before the choice of the auction process was made. Results of the current paper were evoked. The auction format that was effectively used remained secret.

or a fraction of the object for sale² as in the Global One case, we speak of a vertical toehold.³ When a bidder owns shares of another bidder, we speak of an horizontal toehold. We study how both types of toeholds affect the revenue and the efficiency of auction formats in an independent private value setting in the first-price auction and the second-price auction.⁴

The issue of toeholds in an auction context has been the object of few studies. However, some works on different considerations can be understood in terms of toeholds. More specifically, Engelbrecht-Wiggans (1994) can be reinterpreted as a study of an auction with vertical toeholds.⁵ He showed that the expected price is higher with a second-price auction than with a first-price auction.⁶ He considered that this results unbalances Myerson's Revenue Equivalence Theorem (see Myerson (1981)). In fact, proposition 3.5 is a reformulation of this result. However, we show, through our proof of this proposition, that, on the contrary, this result can be interpreted as a consequence of the Revenue Equivalence Theorem. The use of the Revenue Equivalence Theorem even allows to better understand the origins and consequences of the difference between the first-price auction and the second-price auction. Moreover, this paper is the first analyzing the effects of horizontal toeholds. We show that horizontal toeholds are qualitatively different from vertical toeholds. They cannot be interpreted as negative vertical toeholds. Finally, the joint analysis of both types of toeholds allows to have a general interpretation of the results in terms of the sensitivity of auction formats. Let us be more precise and introduce these results.

Firstly, neither vertical toeholds, nor horizontal toeholds, as long as they are symmetric, affect efficiency. Secondly, toeholds do have an impact on the expected revenue of auctions. In both formats, vertical toeholds raise the expected revenue and horizontal toeholds decrease it. Thirdly, these variations are exacerbated under the second-price auction relative to the first-price auction. When bidders

²Both events are conceptually identical.

³A toehold is not consubstantially horizontal or vertical. The context determines whether the toehold is horizontal or vertical.

⁴Here, the two other standard auction formats, the descending and the ascending auction are equivalent to the first-price auction and second-price auction, respectively.

⁵His original motivation comes from Amish estate sales. After the death of a member of the community, the farm is auctioned among the heirs and the resulting revenue is split among them.

⁶This result is also used in Maasland and Onderstal (2002). Independently from this work, they developed a related framework which can also be partially interpreted in terms of vertical toeholds.

have vertical toeholds, the expected revenue is larger under a second-price auction than under a first-price auction. When toeholds are horizontal, the expected revenue is smaller under the second-price auction than under the first-price auction. Therefore, which auction format among the first-price and the second-price auction generates more revenue depends on the type of toehold considered.

The intuition is as follows. When bidders own shares of the seller of the good, they get back, through their shares, a fraction of the amount given to the seller. For both bidders, this is true whether they win or lose the auction. Then, in both auction formats, bidders tend to bid more aggressively and the expected price is higher than without toeholds. However, a losing bidder can have a direct impact on the price only in a second-price auction. Then, bidders have an extra incentive to be aggressive with the second-price auction. That is why the expected price is higher with a second-price auction than with a first-price auction.

When bidders have crossholdings, we observe the opposite phenomenon. If a bidder does not obtain the good, he prefers the price to be low. Suppose that bidder i owns a fraction of bidder j . If bidder i loses the auction, he prefers that bidder j pays the lowest possible price. This gives incentives to bid less aggressively than without toeholds. Consequently, with both auction formats, the expected price is lower than in the standard case without toeholds. Once again, a losing bidder can have a direct impact on the price only in a second-price auction. Because of this more direct leverage, bidders decrease their bids more strongly⁷ in a second-price auction than in a first-price auction. Thus, the expected revenue is lower with the second-price auction than with the first-price auction.

As we already mentioned, the impact of shareholdings has been scarcely discussed in the auction theory literature. Few papers deal with vertical toeholds while horizontal toeholds have been completely ignored. Cramton, Gibbons and Klemperer (1987) study a case of vertical toeholds in which the whole target firm is owned, *ex ante*, by asymmetric competitors. They consider the necessary conditions for the existence of an efficient auction. In their framework, the question of the auction revenue does not make much sense. Once the allocation of the good is determined, there are only transfers between the bidders. There is no actual sale price. In another vein, Singh (1998) and Burkart (1995) analyze a contested takeover in which one bidder owns a vertical toehold in the target firm. He faces a bidder without toehold. They observe overbidding and inefficiencies deriving from the asymmetry among bidders. They do not study the equilibrium of the

⁷In comparison with the bids they would have submitted if there were no horizontal toeholds.

first-price auction. Thus, we do not know if our ranking can be applied to their asymmetric setting. Bulow, Huang and Klemperer (1999) also considers vertical toeholds but in a common value framework. They argue that the common value paradigm is more appropriate to represent **financial** bidders. In that sense, their approach is complementary to ours in which we consider **strategic**⁸ buyers. In their common value framework, they also observe that the revenue of the second-price auction is higher than the revenue of the first-price auction when toeholds are symmetric.

Our approach is also reminiscent of some aspects of the study of auctions with externalities initiated by Jehiel and Moldovanu. In Jehiel and Moldovanu (2000), for instance, they consider an asymmetric information setting in which a losing bidder derives a positive or negative fixed externality from the allocation of the good to another bidder. A priori, our vertical (resp: horizontal) toeholds could be assimilated to a negative (resp: positive) externality except that here what really matters is that the externality term depends on the price. This specific issue does not appear in the auction with externalities literature while our comparison of the first-price auction and the second-price auction crucially relies on it.

The remainder of the paper is organized as follows. In section 2, we present the model. In section 3, we analyze the impact of vertical toeholds. In section 4, we study the horizontal toehold case. Section 5 concludes.

2. The model

A good is sold through an auction process with two risk-neutral bidders, 1 and 2. Bidders 1 and 2 perfectly represent the interests of firms 1 and 2, respectively.⁹ Firm i 's valuation¹⁰ for the good is v_i which is bidder i 's private information. It is common knowledge that v_1 and v_2 are independently distributed according to an identical cumulative distribution F with density f on the interval $[0, 1]$. We further assume that F is continuous, differentiable and strictly increasing on $[0, 1]$.

We define two categories of toeholds. A bidder has an **horizontal toehold** when he owns a fraction of the capital of the other bidder. A bidder has a **vertical toehold** when he owns a fraction of the capital of the seller. For sake of simplicity, we assume that the only asset of the seller is the good for sale. Thus, we can identify

⁸For a discussion on the difference between a strategic and a financial bidder, see the introduction of Bulow, Huang and Klemperer (1999).

⁹In the following of the paper, we will identify bidders with the firms they represent.

¹⁰Throughout paper, by default $i = 1, 2$ and $j \neq i$.

the seller with his good. For a bidder, owning a vertical toehold is equivalent to possessing a fraction of the good for sale.

We focus on two polar cases regarding distribution of toeholds.¹¹

- Vertical toeholds: Both firms own an identical fraction $\alpha \in (0, \frac{1}{2})$ of the capital of the seller and no horizontal toehold.
- Horizontal toeholds: Each firm owns an identical fraction $\theta \in (0, \frac{1}{2})$ of the capital of the other firm and no vertical toehold.

In both cases, toeholds are common knowledge.

We consider two different auction formats: the second-price auction and the first-price auction defined as follows. In both auctions, each bidder simultaneously submits a bid $b \geq 0$ and the bidder who submits the highest bid obtains the good. In the first-price auction, the winner pays the amount of his bid. In the second-price auction, he pays the second highest bid which coincides here with the bid of his opponent. In both auctions, if bidders submit exactly the same bid, the seller flips a fair coin. The winner obtains the good and pays the common bid.

In the vertical toehold case, we suppose that the loser always accepts to sell his fraction of the good at the price defined by the auction.¹² Bidder i owns a fraction α of the good. Then, if he wins the auction, he buys the remaining $(1 - \alpha)$ shares of the good. On the other hand, if he loses the auction, bidder i sells his toehold. Utilities are then defined as follows.

If i obtains the good for a price p :

$$U_i = v_i - p + \alpha p$$

If j obtains the good for a price p :

$$U_i = \alpha p$$

.

¹¹In both cases, we assume that bidders are symmetric. It is a necessary condition to solve equilibria.

¹²Consider, for instance, a contested takeover. In that case, a losing competitor usually prefers not to keep his toehold. The reason is that, if the other bidder takes the control of the target firm, he will probably divert the extra profits he can create. The loser is better off selling his shares before this dilution. On the other side, the winning bidder, in most legislations, cannot refuse to buy his adversary's toehold at the price of the winning tender.

Denoting by q_i the probability that i obtains the good and p the price paid, the expected utility of bidder i can be written:

$$U_i = q_i(v_i - p) + \alpha p.$$

The horizontal toehold case is slightly more complex. We assume that through dividends or the rise of shares' value, any additional profit of a firm goes to its shareholders in proportion to their stakes. If bidder i wins the auction and pays a price p , firm i derives a direct profit from this purchase: $v_i - p$. Consequently, the value of a fraction θ of firm i increases by $\theta(v_i - p)$. Since firm j owns a fraction θ of firm i , whenever firm i wins at the price p , firm j 's value increases by $\theta(v_i - p)$. But, firm i also owns a fraction θ of firm j . Then, because of the increase of firm j 's value, firm i 's value now increases by $\theta^2(v_i - p)$. This mechanism continues *ad infinitum* so that the total increase of firm i 's value is $\sum_{k=0}^{\infty} (\theta^2)^k (v_i - p) = \frac{1}{1-\theta^2} (v_i - p)$. We also obtain that the total increase of firm j 's value is $\sum_{k=0}^{\infty} \theta (\theta^2)^k (v_i - p) = \frac{\theta}{1-\theta^2} (v_i - p)$. Consequently, the total increase of firm i 's value if bidder j buys the good for a price p is $\sum_{k=0}^{\infty} \theta (\theta^2)^k (v_j - p) = \frac{\theta}{1-\theta^2} (v_j - p)$.

Up to a rescaling of payoffs,¹³ utility functions can be defined as follows.

If bidder i obtains the good for a price p :

$$U_i = \frac{1}{1+\theta} (v_i - p)$$

If bidder j obtains the good for a price p :

$$U_i = \frac{\theta}{1+\theta} (v_j - p)$$

Using the same notations as in the vertical toehold case, the expected utility of bidder i is:

$$U_i = \frac{1}{1+\theta} (q_i(v_i - p) + (1 - q_i)(\theta(v_j - p)))$$

We remark that an horizontal toehold cannot be modeled as a negative vertical toehold. As a matter of fact, there is a specific element that does not appear in the vertical toehold case. Here, a losing bidder not only cares about the price paid by the loser, as in the vertical toehold case. He also cares about the valuation for the good of the winning bidder. That is why these two types of toeholds cannot

¹³We multiplied the former expressions by $(1 - \theta)$. With such a normalization, $\forall i = 1, 2$, if bidder i wins the auction, whatever the price paid, p , is, $U_i + U_j + p = v_i$.

be represented by a unique coefficient whose sign would be positive for a vertical toehold and negative for an horizontal toehold.

In both the first-price and the second-price auction and for vertical and horizontal toeholds, we restrict attention to symmetric equilibria.

3. Auctions with vertical toeholds

In this section, we consider the setting with vertical toeholds. Bidders own a fraction of the seller or a fraction of the good for sale. We present the equilibria of the two auction formats and compare expected revenues.

Proposition 3.1. *There is a unique symmetric equilibrium of the second-price auction. For $i = 1, 2$, bidder i with valuation v_i bids $b_V^I(v_i)$ where:*

$$b_V^I(v_i) = v_i + \int_{v_i}^1 \left(\frac{1 - F(t)}{1 - F(v_i)} \right)^{\frac{1}{\alpha}} dt$$

Proposition 3.2. *There is a unique symmetric equilibrium of the first-price auction. For $i = 1, 2$, bidder i with valuation v_i bids $b_V^I(v_i)$ where:*

$$b_V^I(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{\frac{1}{1-\alpha}} dt$$

Proofs: See the Appendix.

Corollary 3.3. *Both auction formats are efficient. Except if $v_i = 1$ or 0 , bidding functions are strictly increasing in α and so are the actual and expected revenues.*

Proof: Efficiency derives from the symmetry of the equilibria and the fact that b_V^I and b_V^I are strictly increasing functions of v_i . The rest is a direct consequence of proposition 3.1 and proposition 3.2. Q.E.D.

In both auction formats, we derive a unique symmetric equilibrium in which bidding functions are increasing in the size of the toehold. The expected price also increases with the size of the toehold and the allocation is efficient. These proprieties are shared by both auction formats. However, vertical toeholds affect bidding behaviors in the two auction formats through two different channels that we will now describe.

Bidders care about the price paid, p , even if they fail to obtain the good since they receive αp in that case. Therefore, in the second-price auction, contrary to

the standard case without toeholds, bidding its own valuation is not a dominant strategy. Bidders tend to bid more than their valuations to raise the price, conditional on losing the auction. On the other side, bidding too high is dangerous. In doing so, a bidder can win and pay a price above his valuation for the good. The equilibrium bid is the result of this trade-off.

For higher values of α , the motivation for raising bids increases since bidders receive a bigger fraction of the price paid conditional on losing the auction. Besides, even if a bidder wins he only buys a fraction $(1 - \alpha)$ of the good. Then, the higher α is, the lower the risk conditional on winning the auction. These two elements go in the same direction and equilibrium bidding functions are strictly increasing in α (except for $v = 1$).

In the first-price auction, losing bids have no effects on the price paid by the winner. There is no direct strategic way for the loser to raise the price paid by the winner. The choice of a bid only fixes the probability of winning and the price paid conditional on winning the auction. However, toeholds have an impact on bidders' incentives, even in a first-price auction.

Each bidder wears two hats: a buyer hat and a seller hat. If he wins, he is the buyer of a fraction $(1 - \alpha)$ of the good. If he loses, he is the seller of a fraction α of the good. As a potential buyer, a bidder, if he bids ε more and wins the auction, does not pay ε more but rather $(1 - \alpha)\varepsilon$ more. Therefore, conditional on winning the auction, bidding more is less costly than without toehold. As a potential seller, the expected utility of bidder i , if he eventually loses the auction while he had submitted a bid d , is:¹⁴ $\alpha \int_{b_j^{-1}(d)}^1 \frac{b_j(t)f(t)}{1-F(b_j^{-1}(d))} dt$. This is strictly increasing in d . By increasing his bid, a bidder reduces his probability to sell his toehold for a low price.

For higher values of α , bidders have two reasons to increase their bids. First, if they lose the auction, they sell a bigger fraction of the good. Thus, it is even more important not to sell for too low a price. Second, if they win, since they buy a smaller fraction $(1 - \alpha)$ of the good, paying a higher price is less costly. In the first-price auction also, equilibrium bids increase in α .

Now, let us reconsider bidders' strategies in the second-price auction. Bidders submit bids above their valuations. As a result, bidder i can win the auction and buy the good for a price strictly above his valuation. It happens whenever $v_i < b^{II}(v_j) < b^{II}(v_i)$. Bidder i wins the auction and pays a price $b^{II}(v_j)$ that is strictly higher than his valuation. As this arises because of his toehold, we

¹⁴At the equilibrium, b_j , the bidding function of j , is a strictly increasing function of v_j .

say that bidder i is victim of the owner's curse.¹⁵ Since the bidder only buys a fraction $(1 - \alpha)$ of the good, he does not always derive a negative utility. However, for some realizations, he may actually derive a strictly negative utility ex post¹⁶ as the following example illustrates.

Example 3.4. *Suppose that F is a uniform distribution, $\alpha = \frac{1}{5}$ and $(v_1, v_2) = (0, \frac{1}{10})$. At the equilibrium, bidder 2 wins the auction, the price is $b^{II}(0) = \frac{1}{6}$ and bidder 2's utility is equal to: $(\frac{1}{10} - \frac{1}{6}) + \frac{1}{5} \times \frac{1}{6} = -\frac{1}{30}$.*

This owner's curse only appears in second-price auctions since in the first-price auction bidders always submit less than their valuations. This seems to indicate that the second-price auction strengthens more competition than the first-price auction does. This intuition is confirmed by the following proposition which establishes the non equivalence in terms of revenue between the two auction formats.

Proposition 3.5. *For any $\alpha \in (0, \frac{1}{2})$, the expected price is strictly higher with a second-price auction than with a first-price auction.*

Proof: The Revenue Equivalence Theorem says that the revenue of an auction is a function of the allocation rule minus bidders' reservation utilities of bidders. In the present case, the allocation is identical with both auction formats. Thus, in order to compare the expected revenues of these two auction formats, we can focus on the comparison of expected utility of lowest type, $v = 0$.

In the first-price auction, the reservation utility is equal to

$$\alpha E(b^I(v_j)) = \alpha \int_0^1 [t - \int_0^t (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}} dx] f(t) dt$$

In the second-price auction, the reservation utility is equal to

$$\alpha b^{II}(0) = \alpha \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt$$

As $\alpha > 0$, comparing these two formulae is equivalent to determining the sign of

$$S_V = \int_0^1 [(1 - F(t))^{\frac{1}{\alpha}} - (t f(t) - \int_0^t f(t) (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}} dx)] dt$$

¹⁵This expression is due to Singh (1998). He used it to describe the same phenomenon in the asymmetric case, with only one bidder owning a toehold.

¹⁶As in the winner's curse case, the ex-post negative utility is only a possibility. For any v_i and α , the expected utility of bidder i is always strictly positive.

Which we can rewrite as:

$$\begin{aligned}
S_V &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 \int_x^1 [1 - (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}}] dx f(t) dt \\
&= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 \int_x^1 [1 - (\frac{F(x)}{F(t)})^{\frac{1}{1-\alpha}}] f(t) dt dx \\
&= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 [F(t) + \frac{1-\alpha}{\alpha} F(x)^{\frac{1}{1-\alpha}} F(t)^{\frac{-\alpha}{1-\alpha}}]_{t=x}^1 dx \\
&= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} dt - \int_0^1 [1 + \frac{1-\alpha}{\alpha} F(x)^{\frac{1}{1-\alpha}} - \frac{1}{\alpha} F(x)] dx \\
S_V &= \int_0^1 (1 - F(t))^{\frac{1}{\alpha}} - 1 + \frac{1}{\alpha} F(t) - \frac{1-\alpha}{\alpha} F(t)^{\frac{1}{1-\alpha}} dt
\end{aligned}$$

As $h(x) = -(1-x)^{\frac{1}{\alpha}} + 1 - \frac{1}{\alpha}x + \frac{1-\alpha}{\alpha}x^{\frac{1}{1-\alpha}} > 0$,¹⁷ for $x \in (0, 1)$, $\alpha \in (0, \frac{1}{2})$, S_V is strictly negative. The reservation utility is higher with a first-price auction than with a second-price auction. Therefore, the expected revenue of the second-price auction is higher than the expected revenue of the first-price auction. Q.E.D.

This result seems to contradict the Equivalence Revenue Theorem of Myerson. But Myerson's Theorem says that for the revenues of two auction formats to be equivalent: (i) The same allocation rule must arise in the two formats. This is clearly met in the present context. (ii) The reservation utility obtained by the lowest bidder must be the same in both auction formats. As the analysis above shows it, this is not so in the present context. The reservation utility is strictly higher with the first-price auction. As a result, the expected revenue is lower with the first-price auction than with the second-price auction.

Now, let us propose an intuitive explanation for this revenue ranking. In the second-price auction, bidders have a more immediate reason to raise their bids. It is a direct way to increase the price conditional on their losing the auction. By the very definition of auction formats, this motivation cannot appear in a first-price auction. That is why vertical toeholds affect more directly bidding strategies in the second-price auction than in the first-price auction. As a result, the expected price is higher with the second-price auction. This format is more sensitive to vertical toeholds.

We established what is the preferred format for the seller. We may also consider bidders' preferences regarding auction formats.

¹⁷This can be demonstrated showing that $\forall \alpha \in (0, \frac{1}{2})$, $h(x)$ on the considered interval is convex then concave and convex. Besides $h(0) = h(1) = 0$ and $h'(0) \geq 0, h'(1) \leq 0, \forall \alpha \in (0, \frac{1}{2})$.

Corollary 3.6. For any $\alpha \in (0, \frac{1}{2})$, $i = 1, 2$ and $v_i \in [0, 1]$, the expected utility of bidder i with a first-price auction is higher than his expected utility with a second-price auction by an amount D_V that is independent of his valuation v_i :

$$D_V = \int_0^1 \alpha(1 - F(t))^{\frac{1}{\alpha}} - \alpha + F(t) - (1 - \alpha)F(t)^{\frac{1}{1-\alpha}} dt > 0$$

Proof: The expected utility of a bidder, whatever his valuation is, is a function of the allocation plus his reservation utility (Revenue Equivalence Theorem). Here, the allocation rule is the same with both auction formats. Therefore, whatever v_i , for bidder i , the difference between his expected utility in the two auction formats is the same, equal to what we compute for $v_i = 0$. Q.E.D.

This uniformity result, although it relies on standard regularity properties, is not so intuitive. It means that the preference of a bidder for an auction format over another one depends on α , the size of his toehold but not on his valuation. The use of a first-price auction rather than a second-price auction is worth a fixed amount to a bidder which does not change with his valuation. If a bidder has a high valuation and he is almost sure to win, the extra utility that he derives from the choice of a first-price auction rather than a second-price auction is the same as when he has a low valuation and he is almost sure to lose the auction. In the first case, he gets an extra utility because he pays a lower price. In the second case, he gets an extra utility because his opponent pays a higher price. Corollary 3.6 tells us that the two effects perfectly compensate for each other.

Now, we also observe that for $\alpha = \frac{1}{2}$, bidders' expected utilities are identical with both auction formats since for any $x \in [0, 1]$, $-(1-x)^{\frac{1}{\alpha}} + 1 - \frac{1}{\alpha}x + \frac{1-\alpha}{\alpha}x^{\frac{1}{1-\alpha}} = 0$ whenever $\alpha = \frac{1}{2}$.¹⁸ In fact, this case corresponds to the partnership problem studied by Cramton and al (1987) showing that the first-price and the second-price auction is equivalent. Besides, it is a well-known result that in the standard case, for $\alpha = 0$, bidders' expected utilities are the same with both auction formats. From these results, we derive the following remark.

Remark 1. For any $i = 1, 2$ and $v_i \in [0, 1]$, the difference between bidder i 's expected utility with a first-price auction and his expected utility with a second-price auction is a non-monotonic function of α .

For low values of α , toeholds do not affect much bidding strategies. Thus, the differences between the two auction formats are minor. For high values of α ,

¹⁸All the former proofs remain true for $\alpha = \frac{1}{2}$.

close to $\frac{1}{2}$, the auction tends to be equivalent to the allocation of a good between his two exclusive owners. In that case, expected utilities uniquely depend on the allocation rule. Since, with both auction formats, the allocation is the same, expected utilities of bidders are also identical with both auction formats. Thus, the choice of an auction format or of the other really matters for bidders only for intermediary values of α .

4. Auctions with horizontal toeholds

In this section, we study the horizontal toeholds case. Two bidders with cross-shareholdings compete in an auction process. We keep the same structure as in the preceding section. That is, we first characterize the symmetric equilibria and then compare the generated revenues.

With horizontal toeholds, incentives are completely opposite to what we have observed with vertical toeholds. If bidder i loses the auction, he has nothing to sell to the winner. In contrast, as he owns a fraction of firm j , he prefers that firm j makes the highest possible profit, that is, he prefers the price to be low.

Thus, for reasons symmetric to those evoked in the case of vertical toeholds, at the equilibrium, bidders submit lower bids than in the standard case. The following propositions show precisely in which way horizontal toeholds affect the equilibrium bidding functions of the two auction formats.

Proposition 4.1. *There is a unique symmetric equilibrium of the second-price auction. For $i = 1, 2$, bidder i bids $b_H^{II}(v_i)$ where:*

$$b_H^{II}(v_i) = v_i - \int_0^{v_i} \left(\frac{1 - F(t)}{1 - F(v_i)} \right)^{\frac{1-\theta}{\theta}} dt$$

Proposition 4.2. *There is a unique symmetric equilibrium of the first-price auction. For $i = 1, 2$, bidder i bids $b_H^I(v_i)$ where:*

$$b_H^I(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{1-\theta} dt$$

Proofs: see the Appendix.

Corollary 4.3. *Both auction formats are efficient. Except if $v_i = 0$, bidding functions are strictly decreasing in θ and so are the actual and expected revenues.*

Proof: Efficiency derives from the symmetry of the equilibria and the fact that b_H^I and b_H^I are strictly increasing functions of v_i . The rest is a direct consequence of proposition 4.1 and proposition 4.2. Q.E.D.

Both auction formats are efficient. Equilibrium bids are decreasing functions of θ . This result is not surprising, but it raises an issue concerning competition regulation. Let us illustrate our point quoting the Commission of the European Communities. In a 1990 report (Case N°IV/M.0004 (1990)), it explained that a shareholdings exchange of 25% between two competitors need not be controlled by the competition authority provided that the exchange “does not in itself either give sole control of one party over the other or create a situation of common control” (in application of Council Regulation N° 4064/89, article 3). Here, even if no common decision is taken, crossholdings distort bidders’ behaviors and affect the price. Therefore, such an exchange, because of its possible consequences, should also be controlled by the authorities in charge of market regulation. This point was already established in the context of a Cournot model (see Reynolds and Snapp (1986)). We showed that it remains true with both the first-price auction and the second-price auction.

Now, let us be more explicit about how crossholdings affect the two auction formats.

In the second-price auction, bidding its own valuation is not a dominant strategy. Bidder i , if he submits a bid b_i and fails to win the auction, derives a utility, $\frac{\theta}{1+\theta}(v_j - b_i)$ which is decreasing in b_i . Thus, bidders tend to bid less than their valuations. Nevertheless, an extremely low bid cannot be part of an equilibrium bidding strategy. As a matter of fact, if bidder i was bidding that way, for some values of v_i , he would leave the good to bidder j for a low price while he would be better off bidding just a little bit more and obtaining the good. Bidding strategies, in the second-price auction, are the result of this trade-off. As θ grows, bidder i gets back a bigger fraction of bidder j ’s profit. Thus, it becomes more and more important for bidder i , conditional on losing, to submit a low bid. That is why the equilibrium bidding function is decreasing in θ .

In the first-price auction, losing bids do not fix the price. Nevertheless, toeholds still affect bidders’ strategies. As a matter of fact, if a bidder does not obtain the good, his utility is the profit of the other bidder multiplied by $\frac{\theta}{1+\theta}$ and not zero. Losing is not as negative an event as it is in the absence of toehold. The equilibrium bidding function which is the result of a trade-off between the fear to lose and the will to make a higher profit when obtaining the good is consequently affected. *Ceteris paribus*, the expected utility of bidder i , conditional on losing

the auction, is increasing in θ . For larger values of θ , bidders are less eager to win the auction with a small difference between their valuations and their bids. As a result, equilibrium bidding functions are increasing in θ .

Thus, through two different channels, in both auction formats, vertical toeholds have a decreasing effect on the bids. We may also ask whether, as in the vertical toehold case, there is a general ranking in terms of expected revenue between the two auction formats.

Proposition 4.4. *For any $\theta \in (0, \frac{1}{2})$, the expected price is strictly higher with a first-price auction than with a second-price auction.*

Proof: As in the proof of proposition 3.5, we can use the Revenue Equivalence Theorem. The allocation is the same with the first-price auction and the second-price auction. Therefore, we directly focus on the reservation utilities comparison.

In the first-price auction the reservation utility of both bidders is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^I(v_j)] = \frac{\theta}{1+\theta} \int_0^1 (t - b_H^I(t)) f(t) dt$$

In the second-price auction, the reservation utility is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^{II}(0)] = \frac{\theta}{1+\theta} \int_0^1 t f(t) dt$$

$\theta > 0$ and $\int_0^1 b_H^I(t) f(t) dt > 0$, then the reservation utility is strictly higher with the second-price auction than with the first-price auction. As a result, the expected revenue is higher with the first-price auction than with the second-price auction. Q.E.D.

As in the former section, we observe that the equivalence in terms of allocation, even in an independent private value framework, is not sufficient to derive the revenue equivalence of the two auction formats. Here also, reservation utilities differ. A bidder with valuation 0 strictly prefers the second-price auction in which he bids 0 and the actual price paid by the other bidder is zero. Therefore, the expected revenue is lower with a second-price auction.

Even though the revenue ranking is opposite, the intuition of this result is similar to the intuition in the vertical toehold case. As a matter of fact, as in the vertical toehold case, a losing bidder can more directly influence the price in the second-price auction. Thus, the downwards variations of expected revenue due to horizontal toeholds are exacerbated in the second-price auction.

Finally, in the case of horizontal toeholds, a seller should be well advised to choose a first-price auction in order to generate more revenue.

We established what is the preferred format for the seller. We may also consider bidders' preferences regarding auction formats.

Corollary 4.5. *For any $\theta \in (0, \frac{1}{2})$, $i = 1, 2$ and $v_i \in [0, 1]$, the expected utility of bidder i with a second-price auction is higher than his expected utility with a first-price auction by an amount D_H that is independent of v_i :*

$$D_H = \frac{\theta}{1 + \theta} \int_0^1 \left(t - \int_0^t \left(\frac{F(u)}{F(t)} \right)^{1-\theta} du \right) f(t) dt > 0$$

Proof: From the Equivalence Revenue Theorem we derive that in any auction mechanism, the expected utility of a bidder is a function of the allocation rule and his valuation plus his reservation utility. Here, the allocation is the same with both auction formats. Therefore, whatever v_i , for bidder i , the difference between his expected utility in the two auction formats is the same, equal to what we computed in zero. Q.E.D.

Bidders' preferences for an auction format over another one depends on θ , the size of the toehold but not on their valuations. The use of a second-price auction rather than a first-price auction is worth a fixed amount to a bidder. This amount does not change with his valuation. For low valuations, bidders prefer the second-price auction mainly because it reduces the price paid conditional on losing. For high values, they prefer the second-price auction because it reduces the price he pays, if he wins. Corollary 4.5 tells us that the two effects perfectly compensate for each other.

5. Conclusion

In the presence of cross-shareholdings, we showed that an expected revenue maximizer seller is always better off choosing a first-price auction. We can apply this result, for instance, to the RVI/Volvo Trucks case that we presented in the introduction. We derive that for the clients of RVI and Volvo Trucks, among the standard auction formats, the first-price auction was the best choice in order to strengthen the competition among them and to avoid as much as possible the negative effects of crossholdings.¹⁹

¹⁹As far as we know, the format that was effectively used was close to a first-price auction. A priori, it was not chosen because of the considerations we studied.

When bidders own a share of the seller or a fraction of the good for sale, the expected revenue is higher with the second-price auction than with the first-price auction. If we apply this result to the Global One case, we derive that a second-price auction was more in favor of Sprint, the seller and a first-price auction was more in favor of France Télécom and Deutsche Telekom, the buyers.

We may also notice that, in that case, it is not obvious that the seller was the one who chose the auction format. As a matter of fact, the two possible buyers owned 80% of Global One while, the seller owned only 20% of Global One. Therefore, it is unclear who really controlled the agenda.

This remark suggests a natural extension to this work. It would consist in modeling a pre-auction bargaining about the choice of the auction procedure. This may be treated in future work. For the time being, we can make the following remark. We established in corollary 3.6 that bidders' preferences for an auction format over another one do not depend on their valuations. Then, without going more into the details of this pre-auction bargaining, we can say that it will not allow to directly extract information about bidders' valuations.

We can also give a broader interpretation of these results. As a matter of fact, losing bidders may care about the final price in many other situations. We could analyze such situations with our model: α and θ would be, respectively, the coefficient of mutual malevolence and the coefficient of mutual benevolence among bidders. With this interpretation, we could extend the application field of our results. We would derive the following. In order to benefit more from the effects of mutual malevolence among bidders, a seller should choose the second-price auction. On the other side, to protect himself from mutual benevolence among bidders, he should choose a first-price auction.

This interpretation is reminiscent of some insights of the existing auction theory literature, although there exists no general study of this issue. Let us give two examples. First, the collusion situation. In that case, we may consider that bidders are mutually benevolent. It has been shown that collusion is much easier to sustain with a second-price auction than with a first-price auction (see Robinson (1985) on this issue). Second, budget-constrained bidders in two sequential auctions. In the first item auction, the losing bidder prefers that the winning bidder pays a high price. If he spends more in the first auction, this winning bidder will be a less dangerous opponent in the following auction. Thus, we can talk of mutual malevolence in the first auction. Pitchik and Schotter (1988) studied that case, focusing on the revenue of the first auction. They showed that the first-price auction raises a lower revenue than the second-price auction, in the first item

auction, both in theory and in practice.

In these two examples, the revenue ranking follows the same general logic as the one we observed in our model. In the benevolent (resp: malevolent) case, a first-price auction (resp: second-price auction) generates more revenue.²⁰ This tends to show that the results we found should be applicable, more generally, to situations in which a seller has to deal with mutual benevolence or mutual malevolence among bidders. Proving this intuition will be the object of future research.

A. Appendix

A.1. Proof of proposition 3.1

As a first step, we must show that any symmetric equilibrium bidding function of the second-price auction, b must satisfy the following conditions: b is continuous in the interval $[0, 1)$, strictly increasing in the interval $[0, 1]$ and $\lim_{v \rightarrow 1^-} b(v) = 1$.²¹

First, let us prove that b is nondecreasing: if $\underline{v} < \bar{v}$, then $b(\bar{v}) < b(\underline{v})$ is impossible. As a matter of fact, as $b(\underline{v})$ is a best response for a bidder with valuation \underline{v} , a bidder with valuation \bar{v} can profitably deviate by submitting $b(\underline{v})$ rather than $b(\bar{v}) < b(\underline{v})$. Thus, b must be nondecreasing. We can also exclude the possibility that b has an atom (an interval of valuations for which bidder i submits the same bid). As a matter of fact, it is impossible that, an interval of types, bidder 2 prefers to quit simultaneously with bidder 1's atom rather than leave just before or just after.

Now, let us show that b must be continuous on $[0, 1)$. Suppose that b has a gap in $v^* \in (0, 1)$. Since bidders strictly prefer to sell their shares for the highest possible price and b is strictly increasing, there always exist an $\varepsilon > 0$ such that a bidder with valuation $v^* - \varepsilon$ is strictly better off submitting $\frac{\lim b(v^*) + \lim b(v^*)}{2}$ rather than $b(v^* - \varepsilon)$ which means that b is not constitutive of a symmetric equilibrium. Thus, b must be continuous on $(0, 1)$. The continuity in 0 can be proved the same way.

²⁰We would not obtain such results if we assumed that losing bidders derive a fixed externality that only depends on the identity of the loser. Jehiel and Moldovanu (2000) showed that, in that case, there is no such revenue ranking. The revenue ranking relies on bidders caring about the price paid by the winner.

²¹At the equilibrium, the exact value of $b(1)$ has no importance provided that $b(1) \geq 1$. For the sake of simplicity, we will consider that $b(1) = 1$.

Finally, suppose that $\lim_{v \rightarrow 1^-} b(v) \neq 1$. As bidding less than his valuation is a dominated strategy, we must have $\lim_{v \rightarrow 1^-} b(v) > 1$. Since b is increasing, we also have $1 < \lim_{v \rightarrow 1^-} b(v) \leq b(1)$. However, if a bidder has a valuation 1 for the good, he strictly prefers that his opponent wins the auction for a price 1 rather than buying the good for a price exceeding 1. Thus, a bidder whose valuation for the good is 1 would be strictly better off bidding 1 than $b(1) > 1$. We obtain a contradiction, therefore, $\lim_{v \rightarrow 1^-} b(v) = 1$.

Now, consider a bidding function b respecting these conditions. If bidder j bids according to b , it is a dominated strategy for bidder i to bid less than $b(0)$ and he cannot be better off bidding more than $b(1)$ than he would be bidding $b(1)$. Thus, we can restrict bidder i 's strategy to the choice of a $g : [0, 1] \rightarrow [0, 1]$, such that he bids $b(g(v_i))$. Let us define $U_i^{II}(v_i, \mathbf{b}_i)$ as the expected utility of bidder i with valuation v_i bidding $b(\mathbf{b}_i)$. As we can limit our study to the case $\mathbf{b}_i \in [0, 1]$, we obtain the following expression

$$U_i^{II}(v_i, \mathbf{b}_i) = \int_0^{\mathbf{b}_i} (v_i - (1 - \alpha)b(t))f(t)dt + \alpha \int_{\mathbf{b}_i}^1 b(\mathbf{b}_i)f(t)dt$$

We obtain the following necessary and sufficient condition²² for b to be a symmetric equilibrium strategy:

$$\frac{\partial U_i^{II}(v, \mathbf{b}_i)}{\partial \mathbf{b}_i} = 0 \text{ for } \mathbf{b}_i = v_i, \text{ for } v_i \in [0, 1]$$

This can be written:²³

$$v_i f(v_i) - b(v_i) f(v_i) + \alpha(1 - F(v_i))b'(v_i) = 0 \quad (\text{A.1})$$

To solve this differential equation, we note that

$$d[b(v_i)(1 - F(v_i))^{1/\alpha}] = [b'(v_i)(1 - F(v_i))^{1/\alpha} - \frac{1}{\alpha}f(v_i)b(v_i)(1 - F(v_i))^{\frac{1-\alpha}{\alpha}}]dv_i \quad (\text{A.2})$$

Then, from expressions (A.1) and (A.2) we derive

$$d[b(v_i)(1 - F(v_i))^{\frac{1}{\alpha}}] = [-\frac{1}{\alpha}(1 - F(v_i))^{\frac{1-\alpha}{\alpha}}v_i f(v_i)]dv_i$$

²²We can exclude corner solutions.

²³We assume that b^0 is well defined on the considered interval, a condition that is verified at the equilibrium.

$$\int_{v_i}^1 d[b(t)(1 - F(t))^{\frac{1}{\alpha}}] dt = \int_{v_i}^1 -\frac{1}{\alpha}(1 - F(t))^{\frac{1-\alpha}{\alpha}} t f(t) dt$$

Integrating by parts, we obtain

$$-b(v_i)(1 - F(v_i))^{\frac{1}{\alpha}} = -v_i(1 - F(v_i))^{\frac{1}{\alpha}} - \int_{v_i}^1 (1 - F(t))^{\frac{1}{\alpha}} dt$$

And finally

$$b(v_i) = v_i + \int_{v_i}^1 \left(\frac{1 - F(t)}{1 - F(v_i)} \right)^{\frac{1}{\alpha}} dt$$

Q.E.D.

A.2. Proof of proposition 3.2

The reasoning is the same as for the proof of proposition 3.1. For the same reasons, b is continuous and strictly increasing. As it is a dominated strategy for a bidder to bid more than his valuation, we derive $b(0) = 0$ (which replaces the limit condition $\lim_{v \rightarrow 1^-} b(v) = 1$). Without repeating the arguments of the proof of proposition 3.1, we can directly study the expression

$$U_i^I(v_i, \mathbf{b}_i) = F(\mathbf{b}_i)(v_i - (1 - \alpha)b(\mathbf{b}_i)) + \alpha \int_{\mathbf{b}_i}^1 b(t)f(t) dt$$

The equivalent of expressions (A.1) and (A.2) are

$$v_i f(v_i) - b(v_i) f(v_i) - (1 - \alpha) F(v_i) b'(v_i) = 0 \quad (\text{A.3})$$

$$d[b(v_i) F(v_i)^{\frac{1}{1-\alpha}}] = [b'(v_i) F(v_i)^{\frac{1}{1-\alpha}} + \frac{1}{1-\alpha} f(v_i) b(v_i) F(v_i)^{\frac{\alpha}{1-\alpha}}] dv_i \quad (\text{A.4})$$

Then, from equations (A.3) and (A.4) we derive

$$d[b(v_i) F(v_i)^{\frac{1}{1-\alpha}}] = \frac{1}{1-\alpha} v_i f(v_i) F(v_i)^{\frac{\alpha}{1-\alpha}}$$

And with transformations identical to these of the proof of proposition 3.1, we obtain

$$b(v_i) = v_i - \int_0^{v_i} \left(\frac{F(t)}{F(v_i)} \right)^{\frac{1}{1-\alpha}} dt$$

Q.E.D.

A.3. Proof of proposition 4.1

Once more, the reasoning of the proof is the same as for the proof of proposition 3.1. All the first part is equivalent, we can directly study the expected utility

$$U_i^{II}(v_i, \mathbf{b}_i) = \frac{1}{1+\theta} \int_0^{\mathbf{b}_i} (v_i - b(t))f(t)dt + \theta \int_{\mathbf{b}_i}^1 (t - b(\mathbf{b}_i))f(t)dt$$

A necessary and sufficient condition²⁴ for b to be a symmetric equilibrium strategy is

$$\frac{\partial U_i^{II}(v_i, \mathbf{b}_i)}{\partial \mathbf{b}_i} = 0 \text{ for } \mathbf{b}_i = v_i, \forall v_i \in [0, 1]$$

This can be written:²⁵

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{\theta}{1-\theta} (1 - F(v_i)) b'(v_i) = 0$$

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = (1 - F(v_i))^{\frac{1-\theta}{\theta}} \int_0^{v_i} \frac{1-\theta}{\theta} (1 - F(t))^{-\frac{1}{\theta}} f(t) dt$$

Integrating by parts, we obtain

$$b(v_i) = v_i - \int_0^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1-\theta}{\theta}} dt$$

Q.E.D.

A.4. Proof of proposition 4.2

We apply the same arguments as in the proof of proposition 3.2 and directly study the expression

$$U_i^I(v_i, \mathbf{b}_i) = \frac{1}{1+\theta} (F(\mathbf{b}_i)(v_i - b(\mathbf{b}_i)) + \theta \int_{\mathbf{b}_i}^1 (t - b(t))f(t)dt)$$

The first-order condition is

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1}{1-\theta} F(v_i) b'(v_i) = 0$$

²⁴We can exclude corner solutions.

²⁵We assume that b^0 is well defined on the considered interval, a condition that is verified at the equilibrium.

As $b(0) = 0$, the solution of the differential equation is

$$b(v_i) = \lim_{x \rightarrow 0} \exp\left[\int_x^{v_i} -(1-\theta) \frac{f(t)}{F(t)} dt \right] \exp\left[\int_0^{v_i} \int_0^t (1-\theta) \frac{f(s)}{F(s)} ds \right] (1-\theta) \frac{f(t)}{F(t)} t dt$$

Integrating by parts, we obtain

$$b(v_i) = v_i - \int_{v_i}^1 \left(\frac{F(t)}{F(v_i)} \right)^{1-\theta} dt$$

Q.E.D.

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