Games with Type-Indeterminate Players

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Abstract

The Type Indeterminacy model is a theoretical framework that uses some elements of quantum formalism to model the constructive preference perspective suggested by Kahneman and Tversky. In this paper we extend the TI-model from simple to strategic decision-making. In Part I we introduce TI-games by means of an example. A 2X2 game is investigated. We let the game be preceded by a cheap-talk promise exchange game. We show in a numerical example that in the TI-model the promise stage can have impact on next following behavior when the standard classical model predicts no impact whatsoever. The TI approach differs from other behavioral approaches in identifying the source of the effect of cheap-talk promises in the intrinsic indeterminacy of the players’ type. In Part II, we formulate some basic concepts for the analysis of games with type indeterminate players. We develop the theory in close connection with the standard approach to game of incomplete information à la Harsanyi. We show an equivalence between static games of incomplete information and static TI-games. We extend this equivalence result to dynamic commuting TI-games. Finally, we develop a new solution concept for non-commuting dynamic TI-games. It differs from the Perfect Bayesian equilibrium by one single property characterizing the updating rule. This property captures the novelty brought about by Type Indeterminacy namely that in addition to affecting payoffs, the action of a player impacts on the profile of types and thus on future actions. We provide an example showing that strategies that form a Perfect Bayesian equilibrium are not part of any Perfect TI-equilibrium.

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1 Introduction

This paper belongs to a very recent and rapidly growing literature where formal tools of Quantum Mechanics are proposed to explain a variety of behavioral anomalies in social sciences and in psychology (see e.g., [9], [1], [2], [4], [7], [11], [8], [18]).

The use of quantum formalism in game theory was initiated by Eisert et al. [10] who study how the extension of classical moves to quantum ones can affect the analysis of a game.4 Another example is La Mura [19] who investigates correlated equilibria with quantum signals in classical games. Whether and when the use of quantum strategies (or strategies using quantum signals) can bring something truly novel to game theory has been discussed in Levine [21] and in Brandenburger [5]. Our approach is different from the so-called quantum game approach. It is based on the idea that players' preferences (types) (rather than the strategies they can choose) can feature non-classical (quantum) properties. We extend of the Type Indeterminacy (TI) model of decision-making introduced by Lambert-Mogiliansky, Zamir and Zwirn [18] from simple decisions to strategic decisions. The TI-model has been proposed as a theoretical framework for modelling the KT(Kahneman–Tversky)–man, i.e., for the "constructive preference perspective".5 Although most of the critic developped by Levine does not apply, it remains an open question as to whether Type Indeterminacy brings fundamentally novel elements to game theory, i.e., that cannot be obtained with the classical tools of game theory. This issue is closely related to the hidden variable argument in Physics.6

A main interest with TI-game is that the Type Indeterminacy hypothesis may modify the way we think about games. Indeed, a major implication of the TI-hypothesis is to extend the field of strategic interactions. This is because actions impact not only on the payoffs of other players but also on the profile of types of the players i.e., who the players are. In a TI-model, players do not have a deterministic, exogenously given, type (preferences). The types change along the game together with the chosen actions (which are modelled as measurements of the type). We illustrate this in an example showing that an initially non-cooperative player can be (on average) turned into a rather cooperative one by confronting him with a "tough" player in a pre-play cheap-talk promise game. We also show that as a consequence of

4From a game-theoretical point of view the approach consists in changing the strategy spaces, and thus the interest of the results lies in the appeal of these changes.

5 There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed – not merely revealed – in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice". (Kahneman and Tversky 2000).

6 As well-known when no restriction are put on the hidden variables all quantum phenomena can be reproduced. But this is no longer true as soon as one requires some desirable properties e.g., locality see [?] for a discussion in the context of quantum games.
strategies that form a Perfect Bayesian equilibrium need not be part of a Perfect TI-equilibrium.

The paper is divided into two parts. In part 1 we introduce some central issues of TI-games. For this purpose we investigate, in two different settings, a 2x2 game with options, to cooperate and to defect and we refer to it as a Prisoner Dilemma, PD. In the first setting, the players move simultaneously and the game is played once. In the second setting, the simultaneous move PD game is preceded by a promise exchange game. One objective is to illustrate how the TI approach can provide an explanation as to why cheap talk promises matter. There exists a substantial literature on cheap talk communication games (see for instance [16] for a survey). The approach in our paper does not belong to the literature on communication games. The cheap talk promise exchange stage is used to illustrate the possible impact of pre-play interaction. Various behavioral theories have also been proposed to explain the impact of cheap talk promises when standard theory predicts that there is none. They most often rely on very specific assumptions amounting to adding ad-hoc elements to the utility function (a moral cost for breaking promises) or emotional communication [13]. Our approach provides an explanation relying on a fundamental structure of the model i.e., the quantum indeterminacy of players’ type. An advantage of our approach is that the type indeterminacy hypothesis also explains a variety of other so called behavioral anomalies such as framing effects, cognitive dissonance [18], the disjunction effect [3] or the inverse fallacy [12].

In part II we develop basic concepts and solutions of maximal information TI-game. We find that much of conventional game theory can be maintained. In particular the standard equilibrium concepts are applicable with non substantial modifications in static games and games where the player only move once (simple signaling games). The crucial novelty appears in multi-stage non-commuting games and it is linked to updating. We formulate a property of the belief function that captures the distinction between the two approaches. We show that this property gives new content (beyond the informational one) to pooling respectively separating behavior. In particular, this allows for a player’s action to impact upon the future type of his opponent. This enlarges the scope of strategic interaction. The intuition is that when the best-reply to an action implies some pooling, some indeterminacy is preserved and the probabilities for next-following choices will be marked by interference effects which are the signature of indeterminacy. In constrast when a best-reply to an action separates between the types of the opponent, interference effect are destroyed. We define a Perfect TI-game equilibrium and demonstrate in an example that the set of Perfect Bayesian equilibrium and Perfect TI-game equilibria do not coincide.

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7 This is for convenience, as we shall see that the game is not perceived as a true PD by all possible types of a player.
8 Cheap talk promises are promises that can be broken at no cost.
Part I
Type Indeterminacy in Strategic Decision-making: An Example

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2 A TI-model of strategic decision-making

In the TI-model a simple "decision situation" is represented by an observable¹¹ called a DS. A decisionmaker is represented by his state or type. A type is a vector \(|t_i|\) in a Hilbert space. The measurement of the observable corresponds to the act of choosing. Its outcome, the chosen item, actualizes an eigentype¹² of the observable (or a superposition¹³ of eigentypes if the measurement is coarse). It is information about the preferences (type) of the agent. For instance consider a TI-model where the agent has preferences over sets of three items, i.e. he can rank any 3 items from the most preferred to the least preferred. Any choice experiment involving three items is associated with six possible eigentypes corresponding to the six rankings of the items. The act of choosing is modelled as a measurement of the (preference) type of the agent and it impacts on the type i.e., it changes it. Suppose an agent in a fully indeterminate state,¹⁴ is confronted with the DS corresponding to "choosing one element out of the \{a, b, c\}" and he chooses a. His type is then projected onto some superposition of the rankings \([a > b > c]\) and \([a > c > b]\). Because the TI-model allows for DS that do not commute, the change in type has implications that go beyond classical information updating. For a detailed exposition of the TI-model see [18]. How does this simple scheme change when we are dealing with strategic decision-making?

We denote by GS (for Game Situation) an observable that measures the type of an agent in a strategic situation, i.e., in a situation where the outcome of the choice, in terms of the agent’s utility, depends

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¹¹An observable is a linear operator that operates on the state of a system.
¹²The eigentypes are the types associate with the eigenvalues of the observable i.e., the possible outcomes of the measurement of the DS.
¹³A superposition is a linear combination of the form \(\sum \lambda_i |t_i|\); \(\sum \lambda_i^2 = 1\) where the \(t_i\) are possible states/types of the players.
¹⁴A state where all of the six potential rankings have positive coefficient of superposition.
on the choice of other agents as well. The interpretation of the outcome of the measurement is that the
chosen action is a best reply against the opponents' expected action. This interpretation parallels the one
in the simple decision context. There, we interpret the chosen item as the preferred one in accordance
with an underlying assumption of rationality i.e., the agent maximizes his utility (he chooses what he
prefers). The notion of revealed preferences (we shall use the term "actualized" rather than revealed15)
and a fortiori of "actualized best-reply" is problematic however. A main issue here is that a best-reply
is a response to an expected play. When the expected play involves subjective beliefs there may be a
problem as to the observability of the preferences. This is in particular so if subjective beliefs are quantum
properties.16 But in the present context of maximal information games (see below for precise de

finition) we are dealing objective probabilities so it is warranted to talk about actualized best-reply.

TI-games are games with type indeterminate players, i.e., games characterized by uncertainty. In par-
ticular, players do not know the payoff of other players. The standard (classical) approach to incomplete
information in games is due to Harsanyi. It amounts to transforming the game into a game of imperfect
information where Nature moves at the beginning of the game and selects, for each player, one among a
multiplicity of possible types (payoff functions). A player’s own type is his private information. But in a
TI-game the players may not even know their own payoff. This is true even in TI-game of maximal infor-
mation where the initial types are pure types.17 Can the Harsanyi approach be extended to TI-games?
We shall argue that the TI-paradigm gives new content to Harsanyi’s approach. What is a fictitious
Nature’s move in Harsanyi’s setting becomes a real move (a measurement) with substantial implications.
And the theoretical multiplicity of types of a player becomes a "real" multiplicity of "selves".

Types and eigentypes We use the term type to refer to the quantum pure state of a player. A pure
type is maximal information about the player i.e., about his payoff function. But because of (intrinsic)
indeterminacy, the type is not complete information about the payoff function in all games simultaneously
not even to the player himself (see [7] for a systematic investigation of (non-classical) indeterminacy with
application to Social Sciences).

15The expression revealed preferences implicitly assumes that the preferences pre-existed the measurement and that
they are uncovered by the measurement. A central feature of the TI-model is precisely to depart from that assumption.
Preferences do not pre-exist the measurement. Preferences are in a state of potentials that can be actualized by the
measurement.
16If subjective beliefs and preferences are quantum properties that do not commute then they cannot be measured
simultaneously.
17Pure types provide maximal information about a player. But in a context of indeterminacy, there is an irreducible
uncertainty. It is impossible to know all the type characteristics of a player with certainty. For a discussion about pure and
mixed types see Section 3.2 in [7].
In a TI-game we also speak about the eigentypes of any specific game $M$, these are \textit{complete information} about the payoff functions in a specific static game $M$. Any eigentype of a player knows his own $M$-game payoff function but he may not know that of the other players. The eigentypes of a TI-game $M$ are identified with their payoff function in that game.

So we see that while the Harsanyi approach only uses a single concept, i.e., that of type and it is identified both with the payoff function and with the player. In any specific TI-game $M$, we must distinguish between the type which is identified with the player and the eigentypes (of $M$) which are identified with the payoff functions in game $M$. A helpful analogy is with multiple-selves models (see e.g., [22] and [14]). In multiple-selves models, we are most often dealing with two "levels of identity". These two levels are identified with short-run impulsive selves on the one side and a long-run "rational self" on the other side. In our context we have two levels as well: the level of the player (the type) and the level of the selves (the eigentypes) which are to be viewed as potential incarnations of the player \textit{in a specific game}.

A central assumption that we make is that the reasoning leading to the determination of the best-reply is performed at the level of the eigentypes of the game. This key assumption deserves some clarification. What we do is to propose that players are involved in some form of parallel reasoning: all the active (with non-zero coefficient of superposition) eigentypes perform their own strategic thinking. Another way to put it is that we assume that the player is able to reason from different perspectives. Note that this is not as demanding as it may at first appear. Indeed we are used in standard game theory to the assumption that players are able to put themselves "in the skin" of other players to think out how those will play in order to be able to best-respond to that.

As in the basic TI-model, the outcome of the act of choosing, here a move, is information about the (actualized) type of the player and the act of choosing modifies the type of the player e.g., from some initial superposition it "collapses" onto a specific eigentype of the game under consideration (see next section for concrete examples).

Finally, we assume that each player is an independent system i.e., there is no entanglement between players.\footnote{In future research we intend to investigate the possibility of entanglement between players.}

We next investigate an example of a maximal information two-person game. The objective is to introduce some basic features of TI-games in a simple context and to illustrate an equivalence and some distinctions between the Bayes-Harsanyi approach and the TI-approach.
2.1 A single interaction

Consider a 2X2 symmetric game, $M$, and for concreteness we call the two possible actions cooperate (C) and defect (D) (as in a Prisoner’s Dilemma game but as we shall see below for certain types, it is a coordination game) and we define the preference types of game $M$ also called the $M$-eigentypes as follows:

$\theta_1$ : prefers to cooperate whatever he expects the opponent to do;
$\theta_2$ : prefers to cooperate if he expects the opponent to cooperate with probability $p > q$ (for some $q \leq 1$) otherwise he prefers to defect;
$\theta_3$ : prefers to defect whatever he expects the opponent to do.

An example of these types is in the payoff matrices below where we depict the row player’s payoff:

$\theta_1$ : 
\[
\begin{pmatrix}
  C & D \\
  C & 10 & 5 \\
  D & 0 & 0 \\
\end{pmatrix}
\]

$\theta_2$ : 
\[
\begin{pmatrix}
  C & D \\
  C & 10 & 0 \\
  D & 6 & 8 \\
\end{pmatrix}
\]

$\theta_3$ : 
\[
\begin{pmatrix}
  C & D \\
  C & 0 & 0 \\
  D & 10 & 5 \\
\end{pmatrix}
\]

We shall now proceed to investigate this simultaneous move TI-game. We note immediately that $\theta_1$ and $\theta_3$ are non-strategic while $\theta_2$ is, i.e., his best-reply will depend on what he expects the opponent to do. The initial types are generally not eigentypes of the game under consideration. Let player 1 be described by the superposition

\[|t_1\rangle = \lambda_1 |\theta_1\rangle + \lambda_2 |\theta_2\rangle + \lambda_3 |\theta_3\rangle, \sum \lambda_i^2 = 1. \tag{1} \]

We shall first be interested in the optimal play of player 1 when he interacts with a player 2 of different eigentypes. Suppose he interacts with a player 2 of eigentype $\theta_1$. Using the definitions of the eigentypes $\theta_i$ above and (1), we know by Born’s rule\(^{19}\) that with probability $\lambda_1^2 + \lambda_2^2$ player 1 plays C (because $\theta_2$’s best-reply to $\theta_1$ is C) and he collapses on the (superposed) type $|t'_1\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\theta_1\rangle + \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\theta_2\rangle$. With probability $\lambda_3^2$ player 1 plays D and collapses on the eigentype $\theta_3$. If instead player 1 interacts with a player 2 of type $\theta_3$ then with probability $\lambda_3^2$ he plays $C$ and collapses on the eigentype $\theta_1$ and since $\theta_3$’s best-reply to $\theta_3$ is $D$, with probability $\lambda_3^2 + \lambda_3^2$ he plays $D$ and collapses on type $|t''_1\rangle = \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_3^2}} |\theta_2\rangle + \frac{\lambda_3}{\sqrt{\lambda_3^2 + \lambda_3^2}} |\theta_3\rangle$.

We note that the probabilities for player 1’s moves depends on the opponent’s type and corresponding expected play - as usual. More interesting is that, as a consequence, the resulting type of player 1 also depends on the type of the opponent. This is because in a TI-model the act of choice is a measurement.

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\(^{19}\)The calculus of probability in Quantum Mechanics is defined by Born’s rule according to which the probability for the different eigentypes is given by the square of the coefficients of superposition.

\(^{20}\)As in the original TI-model, the coefficients of superposition are real numbers and not complex numbers as in Quantum Mechanics. We motivation for that can be found in ??.
that operates on the type and changes it. We interpret the resulting type as the initial type modified by the measurement. In a one-shot context, this is just an interpretation since formally it cannot be distinguished from a classical informational interpretation where the resulting type captures our revised beliefs about player 1 (when our initial beliefs are given by (1)).

We now consider a case when player 2’s type is indeterminate as well:

\[ t_2 = \gamma_1 |\theta_1| + \gamma_2 |\theta_2| + \gamma_3 |\theta_3|, \sum \gamma_i^2 = 1. \]  

(2)

From the point of view of the eigentypes of a player (the \( \theta_i \)), the situation can be analyzed as a standard situation of incomplete information. We consider two examples:

**Example 1** Let \( \lambda^2_1 \geq q \), implying that the eigentype \( \theta_2 \) of player 2 cooperates and let \( \gamma^2_1 + \gamma^2_2 \geq q \) so the eigentype \( \theta_2 \) of player 1 cooperates as well.

**Example 2** Let \( \lambda^2_1 \geq q \) so the eigentype \( \theta_2 \) of player 2 cooperates but now let \( \gamma^2_1 + \gamma^2_2 < q \) so here the eigentype \( \theta_2 \) of player 1 prefers to defect.

In Example 1 the types \( \theta_1 \) and \( \theta_2 \) of both players pool to cooperate. So in particular player 1’s resulting type is a superposition of \( |\theta_1| \) and \( |\theta_2| \) with probability \( (\lambda^2_1 + \lambda^2_2) \) and it is the eigentype \( |\theta_3| \) with probability \( \lambda^2_3 \). In Example 2, player 1’s eigentypes \( \theta_2 \) and \( \theta_3 \) pool to defect so player 1’s resulting type is a superposition of \( |\theta_2| \) and \( |\theta_3| \) with probability \( \lambda^2_2 + \lambda^2_3 \) and \( |\theta_4| \) with probability \( \lambda^2_4 \). So we see again how the resulting type of player 1 varies with the initial (here superposed) type of his opponent.

**Definition**

A pure static TI-equilibrium of a game \( M \) with eigentypes \( \theta_i \in E_1 = E_2 = E \), with action set \( A = \{a_1, a_2\} \) and strategy sets \( S_1 = S_2 = S : E \rightarrow S \) and initial types \( (|t_1^0|, |t_2^0|) \) is

i. A profile of pure strategies \( (s^1_1, s^2_2) \in S \times S \) such that each one of the \( M \)-eigentypes of each player maximizes his expected utility given the (superposed) type of his opponent and the strategies played by the opponent’s eigentypes:

\[ s^1_i(\theta^1_i) = \arg \max_{s^1_i \in S^1_i} \sum_{\theta^1_i; \gamma_i > 0} \gamma_i^2 u_i (s^1_i, s^2_i(\theta^2_i), (\theta^1_i, \theta^2_i)) \]  

(3)

and similarly for player 2.

ii. A corresponding profile of resulting types, one for each player and each action: A corresponding profile of resulting types \( (t'_1, t'_2) \), one for each player

\[ |t'_1| a_i = \sum_{e_i, i^1(e_i) = a_i^1} \lambda^1_i |\theta^1_i| \]
where $\lambda'_i = \frac{\lambda_i}{\sqrt{\sum_j \lambda_j^2}}$ and $a_i$ is the action played by player 1. Similarly for $|t'_2 a_2^2\rangle$.

For concreteness we shall now solve for the TI-equilibrium of this game for the following initial types

$|t_1\rangle = \sqrt{.7} |\theta_1\rangle + \sqrt{.2} |\theta_2\rangle + \sqrt{.1} |\theta_3\rangle$, \hspace{1cm} (4)

$|t_2\rangle = \sqrt{.2} |\theta_1\rangle + \sqrt{.6} |\theta_2\rangle + \sqrt{.2} |\theta_3\rangle$. \hspace{1cm} (5)

Given the payoff matrices above, the threshold probability $q$ that rationalizes the play of $C$ for the eigentype $\theta_2$ is $q = .666$. For the ease of presentation, we let $q = .7$. We know that the $\theta_2$ of player 2 cooperates since $\gamma_1^2 + \gamma_2^2 > q$.

In the TI-equilibrium of this game player 1 plays $C$ with probability $.9$ and collapses on $|t'_1\rangle = \frac{\sqrt{.7}}{\sqrt{.7} + .2} |\theta_1\rangle + \frac{\sqrt{.2}}{\sqrt{.7} + .2} |\theta_2\rangle$ and with probability $.1$ player 1 plays $D$ and collapses on $|\theta_3\rangle$. Player 2 plays $C$ with probability $.8$ and collapses on $|t'_2\rangle = \frac{\sqrt{.8}}{\sqrt{.8} + .4} |\theta_1\rangle + \frac{\sqrt{.4}}{\sqrt{.8} + .4} |\theta_2\rangle$ and with probability $.2$, he plays $D$ and collapses on $|\theta_3\rangle$.

We note that the mixture actually played by player 1 (.9C, .1D) is not the best reply of any of his eigentypes. The same holds for player 2. The eigentypes are the "real players" and they play pure strategies.

We end this section with a comparison of the TI-game approach with the standard incomplete information treatment of this game where the square of the coefficients of superposition in (1) and (2) are interpreted as players’ beliefs about each other. The sole substantial distinction is that in the Bayes-Harsanyi setting the players privately learn their own type before playing while in the TI-model they learn it in the process of playing. A player is thus in the same informational situation as his opponent with respect to his own play. However under our assumption that all the reasoning is done by the eigentypes, the classical approach and the TI-approach are indistinguishable. They yield the same equilibrium outcome. The distinction is merely interpretational.

**Statement 1**

The equilibrium predictions TI-model of a simultaneous one-move game are the same as those of the corresponding Bayes-Harsanyi model.

A formal proof of Statement 1 can be found in Part II.

This central equivalence result should be seen as an achievement which provides support for the hypotheses that we make to extend the basic TI-model to strategic decision-making. Indeed, we do want the non-classical model to deliver the same outcome in a simultaneous one-move context.\footnote{We know that quantum indeterminacy cannot be distinguished from incomplete information in the case of a single measurement. A simultaneous one-move game corresponds to two single measurements performed on two non-entangled systems.} We next move...
to a setting where one of the players is involved in a sequence of moves. This is the simplest setting in which to introduce the novelty brought about by the type indeterminacy hypothesis.

2.2 A multi-stage TI-game

In this section we introduce a new interaction involving player 1 and a third player, a promise exchange game.\(^{22}\) We assume that the \(GS\) representing the promise game do not commute with the \(GS\) representing the game \(M\) (described in the previous section).\(^{23}\) Player 1 and 3 play a promise game where they choose between either making a non-binding promise to cooperate with each other in game \(M\) or withholding from making such a promise. Our objective is to show that playing a promise exchange game - with a third player - can increase the probability for cooperation (decrease the probability for defection) between the player 1 and 2 in a next following game \(M\). Such an impact of cheap-talk promises is related to experimental evidence reported in Frank (1988)

We shall compare two situations called respectively protocol I and II. In protocol 1 player 1 and 2 play game \(M\). In protocol II we add a third player, 3, and we have the following sequence of events:

\(\text{step 1}\) Player player 1 and 3 play a promises exchange game \(N\), described below.

\(\text{step 2}\) Player 1 and 2 play \(M\).

\(\text{step 3}\) Player 1 and 3 play \(M\).\(^{24}\)

The promise exchange game

At \(\text{step 1}\), player 1 and 3 have to simultaneously select one of the two announcements: "I promise to play cooperate", denoted, \(P\), and "I do not promise to play cooperate" denoted \(\text{no} - P\). The promises are cheap-talk i.e., breaking them in the next following games has no implications for the payoffs i.e., at step 2 or step 3.

There exists three eigentypes in the promise exchange game:

\(\tau_1\) : prefers to never make cheap-talk promises - let him be called the "honest type";

\(\tau_2\) : prefers to make a promise to cooperate if he believes the opponent cooperates with probability \(p \geq q\) (in which case he cooperates whenever he is of type \(\theta_2\) or \(\theta_1\) or any superposition of the 2). Otherwise he makes no promises - let him be called the "sincere type";

\(^{22}\)The reason for introducing a third player is that we want to avoid any form of signaling. The exercise could be done with only two players but the comparison between the classical and the TI-model would be less transparent.

\(^{23}\)To each game we associate a collection of \(GS\) each of which measures the best reply a possible type of the opponent.

\(^{24}\)The reason why we have the interaction at step 3 is essentially to motivate the promise exchange game. Our main interest will focus on the interaction at step 2.
τ₃ prefers to promise that he will cooperate whatever he intends to do - he can be viewed as the "opportunistic type".

Information assumptions

We make the following assumptions about players' information in the multi-stage game:

i. All players know the statistical correlations (conditional probabilities) between the eigentypes of the two (non-commuting) games.²⁵

ii. At step 2, player 2 knows that player 1 has interacted with player 3 but he does not know the outcome of the interaction.

We note that ii. implies that we are not dealing with an issue of strategic communication between player 1 and 2. No message is being received by player 2.

The classical model

We first establish that in the classical setting we have the same outcome in protocol I and at step 2 of protocol II. We already know from Statement 1 that the predictions of a TI model of game M are the same as the prediction of the classical Bayes-Harsanyi model of the corresponding incomplete information game.

We investigate in turn how the interaction between player 1 and 3 at step 1 affects the incentives and/or the information of player 1 and 2 at step 2. Let us first consider the case of player 1. In a classical setting, player 1 knows his own type, so he learns nothing from the promise exchange stage. Moreover the announcement he makes is not payoff relevant to his interaction with player 2. So the promise game has no direct implication for his play with player 2. As to player 2, the question is whether he has reason to update his beliefs about player 1. Initially he knows |t₁| from which he derives his beliefs about player 1's equilibrium play in game M. By our informational assumption (i) he also knows the statistical correlations between the eigentypes of the two games from which he can derive the expected play conditional on the choice at the promise stage. He can write the probability of e.g., the play of D using the conditional probability formula:

\[ p(D) = p(P)p(D|P) + p(no-P)p(D|no-P). \]  

(6)

He knows that player 1 interacted with 3 but he does not know the outcome of the interaction. Therefore he has no new element from which to update his information about player 1. We conclude that the introduction of the interaction with player 3 at step 1 leaves the payoffs and the information in the game M unchanged. Hence, expected behavior at step 2 of protocol II is the same as in protocol I.

²⁵So in particular they can compute the correlation between the plays in the different games.
The TI-model

Recall that the GS representing the promise game do not commute with the GS representing the game $M$. We now write eq.(1) and (2) in terms of the eigentypes of game $N$, i.e., of the promise stage eigentypes:

$$|t_1\rangle = \lambda_1^N |\tau_1\rangle + \lambda_2^N |\tau_2\rangle + \lambda_3^N |\tau_3\rangle \text{ and } |t_3\rangle = \gamma_1^N |\tau_1\rangle + \gamma_2^N |\tau_2\rangle + \gamma_3^N |\tau_3\rangle.$$  

Each one of the $N$–eigentype can in turn be expressed in terms of the eigentypes of game $M$:

$$|\tau_1\rangle = \delta_{11} |\theta_1\rangle + \delta_{12} |\theta_2\rangle + \delta_{13} |\theta_3\rangle$$  

$$|\tau_2\rangle = \delta_{21} |\theta_1\rangle + \delta_{22} |\theta_2\rangle + \delta_{23} |\theta_3\rangle$$  

$$|\tau_3\rangle = \delta_{31} |\theta_1\rangle + \delta_{32} |\theta_2\rangle + \delta_{33} |\theta_3\rangle$$

where the $\delta_{ij}$ are the elements of the basis transformation matrix. Assume that player 3 is (initially) of type $\theta_3$ with probability close to 1, we say he is a "tough" type. We shall investigate the choice of between $P$ and no-$P$ of player 1 i.e., the best response of the eigentypes $\tau_1$ of player 1.

By definition of the $\tau_1$ type, we have that $\tau_1$ always plays no-$P$ and $\tau_3$ always play $P$. Now by assumption, player 3 is of type $\theta_3$ who never cooperates. Therefore, by the definition of $\tau_2$, player 1 of type $\tau_2$ chooses not to promise to cooperate, he plays no–$P$.

This means that at step 1 with probability $\lambda_1^2 + \lambda_2^2$ player 1 plays no–$P$ and collapses on $|\tilde{t}_1\rangle = \frac{\lambda_1^N}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\tau_1\rangle + \frac{\lambda_2^N}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\tau_2\rangle$. With probability $\lambda_3^2$ he collapses on $|\tau_3\rangle$.

We shall next compare player 1’s propensity to defect in protocol I with that propensity in protocol II. For simplicity we shall assume the following correlations: $\delta_{13} = \delta_{31} = 0$, meaning that the honest type $\tau_1$, never systematically defects and that the opportunistic guy $\tau_3$ never systematically cooperates.

Player 1’s propensity to defect in protocol I

We shall consider the same numerical example as before i.e., given by (4) and (5) so in particular we know that $\theta_2$ of player 1 cooperates so $p(D || t_1) = \lambda_3^2$. But our objective in this section is to account for the indeterminacy due to the fact that in protocol I the promise game is not played. We have

$$|t_1\rangle = \lambda_1 |\tau_1\rangle + \lambda_2 |\tau_2\rangle + \lambda_3 |\tau_3\rangle,$$

26 A basis transformation matrix links the eigentypes of the two $GO$ $M$ and $N$:

$$
\begin{pmatrix}
\langle \tau_1 | \theta_1 \rangle = \delta_{11} & \langle \tau_1 | \theta_2 \rangle = \delta_{12} & \langle \tau_1 | \theta_3 \rangle = \delta_{13} \\
\langle \tau_2 | \theta_1 \rangle = \delta_{21} & \langle \tau_2 | \theta_2 \rangle = \delta_{22} & \langle \tau_2 | \theta_3 \rangle = \delta_{23} \\
\langle \tau_3 | \theta_1 \rangle = \delta_{31} & \langle \tau_3 | \theta_2 \rangle = \delta_{32} & \langle \tau_3 | \theta_3 \rangle = \delta_{33}
\end{pmatrix}.
$$
using the formulas in (7) we substitute for the $|\tau_i\rangle$

$$
|t_1\rangle = \lambda_1^\prime (\delta_{11} |\theta_1\rangle + \delta_{12} |\theta_2\rangle + \delta_{13} |\theta_3\rangle) + \lambda_2^\prime (\delta_{21} |\theta_1\rangle + \delta_{22} |\theta_2\rangle + \delta_{23} |\theta_3\rangle)
+ \lambda_3^\prime (\delta_{31} |\theta_1\rangle + \delta_{32} |\theta_2\rangle + \delta_{33} |\theta_3\rangle).
$$

Collecting the terms we obtain

$$
|t_1\rangle = (\lambda_1^\prime \delta_{11} + \lambda_2^\prime \delta_{21} + \lambda_3^\prime \delta_{31}) |\theta_1\rangle + (\lambda_1^\prime \delta_{12} + \lambda_2^\prime \delta_{22} + \lambda_3^\prime \delta_{32}) |\theta_2\rangle +
(\lambda_1^\prime \delta + \lambda_2^\prime \delta_{23} + \lambda_3^\prime \delta_{33}) |\theta_3\rangle.
$$

We know from the preceding section that both $|\theta_1\rangle$ and $|\theta_2\rangle$ choose to cooperate so

$$
p(D|t_1\rangle) = p(|\theta_3\rangle|t_1\rangle).
$$

Using $\delta_{13} = 0$, we obtain the probability for player 1’s defection in protocol I:

$$
p(D|t_1\rangle)_{MN} = (\lambda_2^\prime \delta_{23} + \lambda_3^\prime \delta_{33})^2 = \lambda_2^\prime \delta_{23}^2 + \lambda_3^\prime \delta_{33}^2 + 2\lambda_2^\prime \delta_{23} \lambda_3^\prime \delta_{33}.
$$

\textit{Player 1's propensity to defect in protocol II}

When the promise game is being played, i.e., the measurement $N$ is performed, we can (as in the classical setting) use the conditional probability formula to compute the probability for the play of $D$

$$
p(D|t_1\rangle)_{MN} = p(P) p(D|P) + p(no - P) p(D|no - P) .
$$

Let us consider the first term: $p(P) p(D|P)$. We know that $p(P) = p(|\tau_3\rangle) = \lambda_3^2$. We are now interested in $p(D|P)$ or $p(D|\tau_3\rangle).$ $|\tau_3\rangle$ writes as a superposition of the $\theta_i$ with $\theta_1$ who never defects, $\theta_3$ who always defect while $\theta_2$’s propensity to defect depends on what he expects player 2 to do. We cannot take for granted that player 2 will play in protocol II as he plays in protocol I. Instead we assume for now that eigentype $\theta_2$ of player 2 chooses to cooperate (as in protocol I) because he expects player 1’s propensity to cooperate to be no less than in protocol I. We below characterize the case when this expectation is correct. Now if $\theta_2$ of player 2 chooses to cooperate so does $\theta_2$ of player 1 and $p(D|\tau_3\rangle) = \delta_{33}^2$ so

$$
p(P) p(D|P) = \lambda_3^2 \delta_{33}^2
$$

We next consider the second term of (9). The probability $p(no - P)$ is $(\lambda_1^2 + \lambda_2^2)$ and the type of player 1 changes, he collapses on $|\tilde{\tau}_1\rangle = \frac{\lambda_1^\prime}{\sqrt{(\lambda_1^\prime)^2 + (\lambda_2^\prime)^2}} |\tau_1\rangle + \frac{\lambda_2^\prime}{\sqrt{(\lambda_1^\prime)^2 + (\lambda_2^\prime)^2}} |\tau_2\rangle$. Since we consider a case when $\theta_2$ of
player 1 cooperates, the probability for defection of type $t_1$ is
\[ \left( \frac{\lambda_1^2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right)^2 \delta_{13}^2 + \left( \frac{\lambda_2^2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right)^2 \delta_{23}^2. \]
Recalling that $\delta_{13} = 0$, we obtain that $p(\text{no} - P) p(D|\text{no} - P)$ is equal to
\[(\lambda_1^2 + \lambda_2^2) \left( \frac{\lambda_2^2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right)^2 \delta_{23}^2 = \lambda_2^2 \delta_{23}^2, \]
which gives
\[ p(D||t_1)_{MN} = \lambda_2^2 \delta_{23}^2 + \lambda_3^2 \delta_{33}^2. \] (10)
Comparing formulas in (8) and (10):
\[ p(D||t_1)_{MN} - p(D||t_1)_M = -2\lambda_2^2 \delta_{23} \lambda_3 \delta_{33} \] (11)
which can be negative or positive because the interference terms only involve amplitudes of probability i.e., the square roots of probabilities. The probability to play defect decreases (and thus the probability for cooperation increases) when player 1 plays a promise stage whenever $2\lambda_2^2 \delta_{23} \lambda_3 \delta_{33} < 0$. In that case the expectations of player 2 are correct and we have that the $\theta_2$ type of both players cooperate which we assumed in our calculation above.\(^\text{27}\)

**Result 1:** When player 1 meets a tough player 3 at step 1, the probability for playing defect in the next following M game is not the same as in the M game alone, $p(D||t_1)_M - p(D||t_1)_{MN} \neq 0$.

It is interesting to note that $p(D||t_1)_{MN}$ is the same as in the classical case. It can be obtained from the same conditional probability formula.

In order to better understand our Result 1, we now consider a case when player 1 meets with a "soft" player 3, i.e., a $\theta_1$ type, at step 1.

**The soft player 3 case**

In this section we show that if the promise stage is an interaction with a soft player 3 there is no effect of the promise stage on player 1’s propensity to defect and thus no effect on the interaction at step 2.

Assume that player 3 is (initially) of type $\theta_1$ with probability close to 1. What is the best reply of the $N$-eigentypes of player 1, i.e., how do they choose between $P$ and $\text{no-P}$? By definition we have that $\tau_1$ always plays $\text{no-P}$ and $\tau_3$ always play $P$. Now by the assumption we just made player 3 is of type $\theta_1$ who always cooperates so player 1 of type $\tau_2$ chooses to promise to cooperate, he plays $P$.

\(^{27}\)For the case the best reply of the $\theta_2$ types changes with the performance of the promise game, the comparison between the two protocols is less straightforward.
Comparing with eq. (8) of protocol I we see that here 1’s propensity to defect. When playing the promise exchange game the two types may either separate propensities to defect. When they coexist they interfere positively (negatively) to reinforce (weaken) player the sincere and opportunistic type coexist in the mind of player 1. Both these two types have a positive unchanged, so as in protocol I the
2’s expectation. And the assumption we make is that he believes that player 1’s propensity to defect is 
As we investigate player 1’s
in the determination of his play of D at step 2 then

\[ p(D \mid \hat{t}_1) = \frac{1}{\lambda_2^2 + \lambda_3^2} \left[ \lambda_2^2 \delta_{23}^2 + \lambda_3^2 \delta_{33} + 2 \lambda_2^2 \lambda_3^2 \delta_{23} \delta_{33} \right] \]

The probability for defection is thus

\[ p(D \mid \hat{t}_1)_{MN} = P(t_1) p(D \mid t_1) + p(\hat{t}_1) p(D \mid \hat{t}_1) = 0 + \left( \lambda_2^2 + \lambda_3^2 \right) \frac{1}{\lambda_2^2 + \lambda_3^2} \left[ \lambda_2^2 \delta_{23}^2 + \lambda_3^2 \delta_{33} + 2 \lambda_2^2 \lambda_3^2 \delta_{23} \delta_{33} \right] = \lambda_2^2 \delta_{23}^2 + \lambda_3^2 \delta_{33} + 2 \lambda_2^2 \lambda_3^2 \delta_{23} \delta_{33} . \]

Comparing with eq. (8) of protocol I we see that here

\[ p(D \mid t_1)_{M} = p(D \mid t_1)_{MN} \]

There is NO effect of the promise stage. This is because the interference terms are still present. We note also that player 2 was correct in his expectation about player 1’s propensity to defect.

Result 2

If player 1’s move at step 1 does not separate between the N-eigentypes that would otherwise interfere in the determination of his play of D at step 2 then

\[ p(D \mid t_1) = p(D \mid t_1)_{MN} . \]

Let us try to provide an intuition for our two results. In the absence of a promise stage (protocol I) both the sincere and opportunistic type coexist in the mind of player 1. Both these two types have a positive propensity to defect. When they coexist they interfere positively (negatively) to reinforce (weaken) player 1’s propensity to defect. When playing the promise exchange game the two types may either separate

\[ \text{Recall that } \tau_1 \text{ never defects.} \]
or not. They separate in the case of a tough player 3. Player 1 collapses either on a superposition of the honest and sincere type (and chooses *no-P*) or on the opportunistic type (and chooses *P*). Since the sincere and the opportunistic types are separated (by the first measurement, game *N*) there is no more interference. In the case of a soft player 3 case, the play of the promise game does not separate the sincere from the opportunistic guy, they both prefer *P*. As a consequence the two Neigentypes interfere in the determination of outcome of the next following *M* game as they do in protocol I.

In this example we demonstrated that in a TI-model of strategic interaction, a promise stage does make a difference for players’ behavior in the next following performance of game *M*. The promise stage makes a difference because it may destroy interference effects that are present in protocol I.

Quite remarkably the distinction between the predictions of the classical and the TI-game only appears in the absence of the play of a promise stage (with a tough player). Indeed the probability formula that applies in the TI-model for the case the agent undergoes the promise stage (10) is the same as the conditional probability formula that applies in the standard classical setting.

**The cheap-talk promise paradox**

When promises that have no commitment or informational value affect behavior, we may speak about a cheap talk paradox (with respect to established theory). In particular we may have the case that despite the fact that all types pool to make cheap-talk promises (we only have non-revealing equilibria), they nevertheless affect subsequent play. Our paper does not exactly address this case. This is because on the one hand playing the promise game always separates between the *τ*₁ and *τ*₃. On the other hand the promises are not communicated to player 2. Yet, because the analysis focuses on the separation between *τ*₂ and *τ*₃ (and by its information assumption avoids Bayesian updating with respect to *τ*₁), it suggests two possible explanations for why cheap talk promises may matter:

1. Unobserved separation

Here the idea is that the promise game actually does trigger separation between types (like in protocol 2 with a taught type). Reaching the promise response is more difficult for the reciprocating type *τ*₂ than for the opportunistic *τ*₃. It takes longer time to do the reasoning. The act of playing breaks the indeterminacy of player 1 but that is not observed by player 2, both *τ*₂ and *τ*₃ choose to promise or they choose differently but player 2 does not learn about it. In that case the TI-model’s predictions in the next following PD game with or without a prior promise exchange game are not the same. We have an impact of cheap-talk promises.

2. Observed pooling

The second line of explanation of the paradox follows a different logic. It relies on the observation
that if the observer has the classical model in mind, his predictions are incorrect. When he confronts his predictions in protocol 2 (which are the same as his predictions in protocol 1) with the actual outcome of protocol 2, he notes a difference. This is because simply he did not account for the interference effects. So here the explanation is not that pooling in cheap-talk promises changes behavior but that there is an error in the modeling of the pooling outcome.

2.3 Concluding remarks

In part 1 of this paper we have explored an extension of the Type Indeterminacy model in an example of a 2X2 game that we investigate in two different settings. In the first setting the game is played directly. In the second setting the game is preceded by a promise exchange game. We first find that in a one-shot setting the predictions of the TI-model are the same as those of the corresponding Bayes-Harsanyi game of incomplete information. This is no longer true in the multiple move setting. We give an example of circumstances under which the predictions of the two models are not the same. We show that the TI-model can provide an explanation for why a cheap-talk promises matter. The promise game may separate between potential eigentypes and thereby destroy interference effects that otherwise contribute to the determination of, e.g., the propensity to defect in the next following game.

We next develop some basic concepts and solutions of TI-games.
Part II
Basic Concepts and Solutions of TI-games

Vladimir Danilov\textsuperscript{29}, Ariane Lambert-Mogiliansky\textsuperscript{30}

3 Static maximal information TI-games

TI-games are games with type indeterminate players, i.e., games characterized by uncertainty, players have incomplete information about important aspects of the game. It is therefore natural to look for the classical counter-part in games with incomplete information.

In this section we deal exclusively with static games. Using the terminology introduced in section 2 of Part 1, we denote by $GS$ (for Game Situation) an operator (called observable in Physics) that measures the type of an agent in a strategic situation i.e., in a situation where the outcome of the choice, in terms of the agent’s utility, depends on the choice of other agents as well. The notion of $GS$ applies exclusively to game situations that can be associated with a first-kind measurement.\textsuperscript{31} The outcome of a $GS$ i.e., of a move, is information about the type of the player.\textsuperscript{32} A $GS$ may not give maximal information about the type of the player.\textsuperscript{33} This is the case when some of the possible types pool in their best-reply. Such a $GS$ is a coarse measurement which preserves some indeterminacy with implications beyond the pure informational aspects as we shall see below.

The interpretation of the outcome of the measurement is that the chosen action is a best reply against the opponent’s expected action.\textsuperscript{34} This interpretation parallels the one in the simple decision context.

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\textsuperscript{31}First-kindness is a property of reproducibility. If one performs a measurement on a system and obtains a result, then one will get the same result if one performs again that measurement on the same system immediately after. First-kindness does not entail that the first outcome is obtained when repeating a measurement if other measurements are performed on the system in between.
\textsuperscript{32}We assume throughout that each player is an independent system i.e., there is no quantum entanglement between players.
\textsuperscript{33}In a TI context there is a distinction between maximal and complete information. See the subsection "type an eigentype".
\textsuperscript{34}As discussed in spiegler the notion of revealed preferences and a fortiori of revealed best reply is not a simple one. A main issue here is that a best reply is a response to an expected play. When the expected play involves subjective beliefs which are not observable then there is a problem as to the observability of the preferences behind the best reply. In the context of maximal information games, we are not dealing with subjective beliefs but with the objective probability reflecting the intial pure state.
There, we interpret the chosen item as the *preferred one* in accordance with an underlying assumption of (basic) rationality i.e., the agent maximizes his utility (i.e., chooses what he prefers). As discussed in Spiegler (2006) the notion of revealed preferences (we shall use the term *actualized* rather than *revealed*\(^\text{35}\)) and a fortiori of actualized best reply is problematic. A main issue here is that a best reply is a response to an *expected* play. When the expected play involves subjective beliefs there may be a problem as to the observability of the preferences. This is in particular so if subjective beliefs are quantum properties.\(^\text{36}\) But in the present context of maximal information games (see below for precise definition) we are dealing objective probabilities so it is warranted to talk about actualized best-reply.

**Uncertainty**  When some players do not know the payoff function of other players the game is said of incomplete information. The standard approach, due to Harsanyi, is to transform the game into a game of imperfect information where Nature moves at the beginning of the game and selects, for each player, one among a multiplicity of possible types. A player's own type is his private information. Generally, the Harsanyi type includes a player's information and beliefs. But in this paper we let the term Harsanyi type exclusively refer to the payoff function while information is dealt with separately. This is because we shall be dealing with TI-games of *maximal information* where all players are represented by pure types (eigentype or superposition of eigentypes, see below for a definition).\(^\text{37}\) As we shall see this implies that we are dealing with objective uncertainty. However in such a TI-game some players may not even know their *own payoff function. Can we nevertheless extend the Harsanyi approach to TI-games? We shall argue that the TI-paradigm gives new content to Harsanyi’s approach. What is a fictitious Nature’s move in Harsanyi’s setting becomes a real move (a measurement) with substancial implications. And the theoretical multiplicity of types of a player becomes a substancial multiplicity of "selves".\(^\text{38}\)

**Types and eigentypes**

We use the term *type* as the term *quantum pure state*. A (pure) type \(|t\rangle \in \mathbb{T}\(^\text{39}\)) where \(\mathbb{T}\) is a Hilbert space, is maximal information about the player i.e., about his payoff function.\(^\text{40}\) But because of (intrinsic)

\(^{35}\)The expression revealed preferences implicitly assumes that the preferences pre-existed the measurement and that they are uncovered by the measurement. A central feature of the TI-model is precisely to depart from that assumption. Preferences do not pre-exist the measurement. Preferences are in a state of potentials that can be actualized by the measurement.

\(^{36}\)If subjective beliefs and preferences are quantum properties that do not commute then they cannot be measured simultaneously.

\(^{37}\)For a discussion about pure and mixed types (states) see Section 3.2 in Danilov and Lambert-Mogiliansky 2008.

\(^{38}\)There exists various models of multiple selves in the literature. The original idea is due to Strotz (1956) and is used to model time inconsistent behavior in dynamic utility maximization

\(^{39}\)We use Dirac ket notation \(|\rangle\) to denote a vector in a Hilbert space.

\(^{40}\)We do not include any private information or beliefs in our definition of a type or of an eigentype.
indeterminacy (see below), a type does not provide complete information about the payoff function in all possible GS not even to the player himself.

In a TI-game we also speak about the eigentypes of a game $M$, $e_i(M) \in E(M), E(M) \subset T$. The term eigentype parallels the term eigenstate of a system. It is a state associated with one of the possible eigenvalues of an observable. An eigentype is thus the type associated with one of the possible outcomes of a GS (or more correctly of a complete set of commuting GS associated with a game). The eigentypes are truly private and complete information about the payoff functions in a specific static game $M$. Any eigentype of a player knows his own $M$-game payoff function but he may not know that of the other players. The eigentypes of a TI-game $M$ are identified with their payoff function in that game.

Let us be more precise as to the distinction between the Harsanyi approach to uncertainty and the Type Indeterminacy approach. Harsanyi assumes that the player’s type are independently drawn from a joint distribution $p(\theta_1, ..., \theta_n)$ where $\theta_i \in \Theta$ is some Boolean space with finite number of elements. A Harsanyi type summarizes all possible type characteristics of a player. The joint distribution is common knowledge among all the players. Upon receiving information from Nature about one’s own type, a player updates his probability about others’ type. The conditional probabilities $p(\theta_\cdot; \theta_i)$ constitute the initial beliefs of player $i$ and these beliefs are updated according Bayes’rule along the game in view of new information.

In TI-games, the types $|t_i\rangle$ are (vectors) elements of a (finite dimensional) Hilbert space $T$. In a maximal information game the initial type of the players are given as pure types and they are common knowledge among players. For any static game $M$, a type $t_i$ can be expressed in terms of the eigentypes $|e_i(M)\rangle \in E(M) \subset T$, with $|e_i(M)\rangle \perp |e_j(M)\rangle, i \neq j$. An eigentype summarizes the type characteristics relevant to a particular game. The uncertainty relevant to game $M$ is given by the expression of the initial type $t_i$ in terms of (a superposition) the eigentypes $e_i(M)$. In contrast to a Harsanyi type, an eigentype does not summarize all the possible type characteristics. Consider a multi-move game: $M$ followed by $N$, the TI-model allows for the case case when the type characteristics relevant to $M$ respective $N$ are “incompatible” in the sense that they cannot be revealed (actualized) simultaneously. This is the source of intrinsic indeterminacy i.e., of an uncertainty not due to incomplete information but to the impossibility of actualizing type characteristics belonging to two non-commuting GS. The classical Bayes-Harsanyi model corresponds to the special case of the TI-model when all GS commute. As a consequence of non-commutativity, the players’ type change as the game proceeds.
of opponents’ type does not generally follow Bayes’ rule.\footnote{See Danilov et al. forthcoming in Theory and Decision 2009 for a discussion on updating in a non-classical context}

We see that while the Harsanyi approach only uses a single concept, i.e., that of type and it is identified both with the payoff function and with the player. In any specific TI-game $M$ we must distinguish between the type which is identified with the player and the eigentypes which are identified with the payoff functions in game $M$. A helpful analogy is with multiple-selves models (see e.g., Strotz 1956, Fudenberg and Levine 2006). In multiple-selves models we are dealing with two "levels of identity". Most often these two levels are identified with the short-run impulsive selves on the one side and a long-run rational self on the other side. In our context we have two levels as well: the level of the player (the type) and the level of the selves (the eigentypes) which are to be viewed as potential incarnations of the player in a specific game. In a TI-model a player is described as a superposition of (simultaneous) selves.

**Assumption 1**

In a TI-game all strategic reasoning is done by the eigentypes of the players.

This key assumption deserves some clarification. What we have in mind is that the player performs some form of parallel reasoning, all the active eigentypes perform their strategic thinking. The player experiences himself as truly hesitating/multiple and is able to reason from different perspectives. Note that this is not as demanding as it may at first seem since in game theory players must be able to put themselves "in the skin" of others to think out how those will play and best respond to that.\footnote{Neurological research, e.g., about mirror neurons, suggests that indeed people do have the capacity to experience a variety of perspectives and that this constitutes a crucial element in understanding each other and being able to interact with each other. A pathological case where this faculty is severed is autism.}

**Utility**

In the context of type indeterminacy we have to be careful with the notion of utility. A question arises because the type of the agent changes along the path of actions with the decisions that are made: which type’s utility is it that matters to the agent’s decisions? We shall adopt the following principle: at each step the utility of an action profile is evaluated by the potential current eigentypes. In a multi-stage context, these types are of course well aware that there is a continuation game and they do care about the utility of the player’s future selves. We shall assume that the utility of the future selves is accounted for as evaluated by these future selves. More precisely: in each period each potential eigentype of a player maximizes his current utility plus the expected utility of the resulting type conditional on "surviving" the current stage.\footnote{This means that the resulting type must include that eigentype among the potential eigentypes resulting from the first move. If that was not the case, that eigentype’s utility would not enter the player’s utility.} This approach secures that the current eigentypes internalise the impact of their...
own decision on the future selves of the player. We adopt the convention of giving the same weight to the utility of all relevant types along the path.

### 3.1 Static TI-game equilibrium

The equilibrium concept of static incomplete information game is the Bayesian Equilibrium. It relies on the Harsanyi type approach to uncertainty described above. Let the set of player \( N, N = \{1, 2\} \), the players are indexed by \( j = 1, 2 \). The sets of pure strategies are denoted \( S_j \) (we shall limit ourselves to pure strategies) with \( s_j : \Theta \to A_j \) where \( A_j \) is the action set. The payoff functions are \( u_j (s_1,s_2,\theta^1,\theta^2) \). If player \( i \) knew the strategies of the other players as a function of their type, player \( i \) could use his beliefs \( p (\theta^{-j}; \theta^i) \) to compute his expected payoff. This is the idea behind the concept of Bayesian equilibrium.

A Bayesian equilibrium in a game of incomplete information with a finite number of types \( \theta_i \) for each player \( j \), prior distribution \( p(.) \), and pure strategy spaces \( S_j \), is a Nash equilibrium where each player \( j \) maximizes his utility conditional on \( \theta^i \)

\[
s_j (\theta^j) = \arg \max_{s_j \in S_j} \sum_{\theta^{-j}} p (\theta^{-j} | \theta^j) u_j (s_j', s_{-j} (\theta^{-j}), (\theta^j, \theta^{-j})). \tag{12}
\]

#### A first equivalence result

As we wrote earlier, in a TI-game a player is not identified with his payoff function so when it comes to equilibrium analysis we cannot proceed as in the classical case. Yet, under assumption 1 all the strategic reasoning is done by the eigentypes (the selves) who know their payoff functions just as the players know their own payoff function in the Bayes Harsanyi model. The eigentypes are the real "players" and we shall see that under this assumption a static TI-game looks very much like an incomplete information game and a static TI-equilibrium can be defined as a Nash equilibrium of the original two-player game expanded to the eigentypes of each player.

Let \( A_j \) be a finite set of actions available to player \( j = 1, 2 \). Each player is represented by his type \( |t_j| \in T \). For any game situation \( M \) we have \( E_j (M) \subset T \), where \( E_j (M) = \{ |e^1_j (M) \}, \ldots, |e^k_j (M) \} \) is the set of eigentypes of player \( j \) in \( GS M \). In the static context of this section we can delete the qualifier in parenthesis and write \( |e^1_j \) and \( E_j \). A pure strategy for player 1 is a function \( s_1 \in S_1, s_1 : E^1 \to A_1 \).

The initial type vector of player \( j = 1 \) can be expressed in terms of the eigenvectors of \( GS M \):

\[
|t_{1} \rangle = \sum_{1}^{k} \lambda_{1}^{1} |e_{1}^{1} \rangle, \sum_{1}^{k} \lambda_{1}^{2} = 1. \tag{13}
\]

---

48 The formulation in terms of utility conditional on the type is equivalent to the unconditional formulation because all types have positive probability.
The initial common knowledge beliefs about the eigentypes are given by the types $|t_j\rangle$ according to Born’s rule

$$\text{prob} \left( e_i^j | t_1 \right) = \lambda_i^2.$$  

We call $e_i^j$ a potential eigentype of player 1 iff $\lambda_i > 0$.\(^{49}\)

In the Bayes Harsanyi model, Nature moves first and selects the (eigen)type of each player who is privately informed about it. In a TI-game uncertainty is (partially) resolved by the measurement i.e., the actual act of playing. So before playing the player does not know his own payoff function (i.e., his eigentype) only his initial (superposed) type (eq. 13). However, each one of his selves (we use the terms self and eigentype interchangeably) knows his own payoff function. We assume that the potential selves of a player all have the same information about the opponent’s type. Now if the selves know the strategy of the eigentypes of their opponent, they can compute their expected payoff using the information encapsulated in the (superposed) initial type of his opponent (we assume that the potential selves of a player all have the same information about the opponent’s type).

**Definition**

A pure strategy TI-equilibrium of a two-player static game $M$ with initial types $|t_1\rangle = \sum \lambda_i |e_i^1\rangle$ and $|t_2\rangle = \sum \gamma_i |e_i^2\rangle$ is

i. A profile of pure strategies $(s_1^*, s_2^*)$ with

$$s_i^* (e_i^j) = \arg \max_{s_i \in S_i} \sum_{e_i^j; \gamma_i > 0} \gamma_i^2 u_i \left( s_i^1 (e_i^j), s_i^2 (e_i^j), (e_i^1, e_i^2) \right)$$

(14)

for all $e_i^j; \lambda_i > 0 i = 1,..k$ and similarly for player 2.

ii. A corresponding profile of resulting types $(t_1', t_2')$, one for each player

$$|t_i' | a_i = \sum_{e_i^j; s_i^* (e_i^j) = a_i} \lambda_i' |e_i^j\rangle$$

where $\lambda_i' = \frac{\lambda_i}{\sqrt{\sum_{j \neq i} (\lambda_j')^2 (s_i^* (e_i^j))}}$ and $a_i$ is the action played by player 1. Similarly for $|t_2' | a_1^* \rangle$.

The first part (i) says that each one of the potential eigentypes of each player maximizes his expected utility given the (superposed) type of his opponent and the strategies played by the opponent’s potential eigentypes. It is very similar to the definition of a Bayesian equilibrium strategy profile except that

\(^{49}\)We note that this is equivalent to an incomplete information representation with player 2’s initial beliefs about 1 given by

$$\sum_{i=1}^{k} p (e_i^1 | e_i^2) = p (e_i^1 | e_i^2) = (\lambda_i)^2$$ for all $e_j$.  

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the probabilities for the opponent’s eigentypes are given by the initial superposed type instead of a joint probability distribution.\textsuperscript{50} In the classical case, a player is identified with an eigentype. Hence, maximizing the eigentype’s utility is the same as maximizing the player’s utility. In a TI-game, a player is a superposition of eigentypes. The question may arises as to whether when each of the potential eigentype maximizes his own utility, the utility of the player also is maximized. In a static context, this issue is rather unproblematic. The (expected) utility of the, possibly superposed, resulting type \( |t_i^*| a_i \), is simply a convex combination of the (potential) eigentypes’ utilities which is increasing in all its components.

The second part of the definition (ii) captures the fact that in a TI-game the players’ type is modified by their play. The rule governing the change in the type is given by the von Neuman-Luder’s projection postulate (see below for a detailed exposition). In a static setting it is equivalent to Bayesian updating within the set \( E_j \). We return to this issue in details below in connection with updating in multi-stage games.

**Proposition 1** A pure strategy TI-equilibrium profile of a static maximal information game \( M \) with eigentypes \( (e_1, \ldots, e_k) \) and initial types \( |t_i \rangle = \sum_{j=1}^{k} \lambda_j^i |e_j \rangle \) is equivalent to a Bayesian pure strategy equilibrium profile of a game with type space \( E_j = \{e_1^i, \ldots, e_k^i\} \) and common knowledge beliefs given by the distributions \( p(e_1^i | t_1) = (\lambda_1^i)^2 \) for both players and for the eigentypes of the opponents \( e_i^2, i = 1, \ldots, k_2 \).

The proof follows immediately from the definitions.

In a TI-equilibrium, each eigentype of each player, if called to play, plays a best-reply to the (superposed) type of his opponent. Nature moves at the time of the actual play, selects an eigentype who already knows his best-reply to the type of the other player. From the point of view of an eigentype the situation is equivalent to that of a player in an incomplete information game à la Harsanyi. The eigentype knows his preferences and can compute his best-reply to the unknown eigentype of his opponent. The fact that in the classical setting Nature moves before the beginning of the game while Nature moves at the time of playing in the TI-setting has no implication for equilibrium analysis under the assumption that all the strategic reasoning is done by the eigentypes.

\textsuperscript{50}Note also that the best-reply condition (14) applies to potential eigentypes i.e., with a strictly positive coefficient of superposition in the initial type only (in the standard classical approach all types have positive probability).
4 Multi-stage TI-game

In the classical context, the static Bayesian equilibrium notion extends to multi-move games when no observation is made between the moves. In such a case the different moves can be merged into a compound move. For instance consider a situation where the players must first choose between Right and Left and thereafter without having observed the opponent’s play, choose between Up and Down. This game can just as well be expressed as a static game with action set $A = \{RU, RD, LU, LD\}$. The actions are merged. This is generally not true in a TI-context (see Part 1 for an example where a move matters even when it is not observed).

When it comes to games composed of more than one step for at least one player, the crucial issue for TI-games is whether the corresponding $GS$ commute with each other or not.

Commutativity of $GS$

We say that two $GS$ $M$ and $N$ commute if they share a common set of eigentypes $E = E(M) \otimes E(N)$.\(^{51}\) Another way to express this is to say that the type are separable $|t\rangle = |t\rangle_M \otimes |t\rangle_N$. This is the standard definition of commuting observables. For the case all $GS$ commute, the Type Indeterminacy representation of uncertainty about other players’ type is equivalent the classical Harsanyi representation.\(^{52}\)

Definition

A commuting multi-stage TI-game is such that for each player all the $GS$ he may face (in and out of the equilibrium path) commute with each other. Otherwise we say that the multi-stage game is non-commuting.

If there is no observation between two commuting moves we can merge the two $GS$ into one compound $GS$ with outcome set $(a_i(M), a_i(N))$. We are in a fully classical situation and the static equilibrium concept applies. More generally, i.e., when we have some observation between the moves, commuting multi-stage Ti-games are not distinguishable from classical Bayesian games. We postpone the proof of this statement.

4.1 Simple signaling games

In standard game theory, the notion of multi-stage games includes games where each player only plays once but in a sequence, e.g., simple signaling games. Simple signaling game qualify as multi-stage...\(^{51}\)When the two $GS$ do not commute the tensor product set $E(M) \otimes E(N)$ does not exist.\(^{52}\)For a general result establishing the equivalence of the classical and the non-classical measurement theory in a multiple commuting measurement context see (see ALM, HZ, SZ 2006 (JMP 2009) or Danilov Lambert-Mogiliansky 2008).
games because updating takes place. The corresponding equilibrium concept is that of Perfect Bayesian Equilibrium.

**Updating**

Generally, updating in non-classical systems is a complex issue (see Danilov and Lambert-Mogiliansky, forthcoming Theory and Decision 2009). However, in the case of Hilbert space models, the earlier mentioned von Neumann-Luder postulate tells us exactly where a measurement takes the state and accordingly how information should be updated. Let the initial type vector be an arbitrary

\[ |t_i\rangle = \sum_{j=1}^{k} \lambda_j |e_j^1\rangle \]

Suppose that the measurement of GS M yields action \( a_1^1 \) which is a best-reply against \( t_2 \) for \( e_1^2 \) and \( e_3^2 \) (the GS is a coarse measurement). According the von Neuman-Luder postulate, the initial type vector \( |t_i\rangle \) is projected onto the eigenspace spanned by \( |e_1^1\rangle \) and \( |e_3^1\rangle \). After the play of \( a_1^1 \) player 1’s type is given by

\[ |t'_i\rangle = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_3^2}} |e_1^1\rangle + \frac{\lambda_3}{\sqrt{\lambda_1^2 + \lambda_3^2}} |e_3^1\rangle. \]

The resulting type \( |t'_i\rangle \) is a superposition of the two eigentypes who pooled in choosing \( a_1^1 \). Given this new type, the updated probability for say \( e_1^1 \) is given by Born’s rule to be the square of the coefficient of superposition:

\[ \text{prob}(e_1^1|t'_i) = \left( \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_3^2}} \right)^2 = \frac{\lambda_1^2}{\lambda_1^2 + \lambda_3^2}. \]

As earlier noticed within \( E_1 \) the von Neuman-Luder postulate induces Bayesian updating for appropriately defined initial beliefs.

Consider a signaling two-player TI-game with action sets \( A_1 \) and \( A_2 \). A pure strategy for player 1 who moves first is \( s_1 : E_1 \rightarrow A_1 \), a strategy for player 2 \( s_2 : E_2 \times A_1 \rightarrow A_2 \).

**Definition**

A pure strategy signaling TI-equilibrium with initial types are \( |t_1\rangle = \sum \lambda_i |e_i^1\rangle, |t_2\rangle = \sum \gamma_i |e_i^2\rangle \) and where player 1 moves first to select \( a_1 \) is characterized by:

\[ P1 \ s_1^1(e_1^i) = \arg \max_{a_1^1 \in S_1} \sum_j \gamma_j^2 u_1^1(a_1^1, s_2^i(e_j^2), e_i^1, e_j^2), \text{ for all } e_i^1 \in E_1 \]

\[ \text{In [18] we discuss the behavioral hypothesis behind our interpretation of pooling in choice as a the outcome of a coarse measurement.} \]
\[ P^2 s^*_2 (e^*_2) = \arg \max_{e^*_1 \in S_2} \sum_{e^*_1} \mu (e^*_1 | a_1) w^2 (s^*_2, a_1, e^*_2, e^*_1) \text{ for all } e^*_2 \in E_2 \]

\[ B \mu (e^*_1 | a_1) = \frac{\lambda^2}{\sum_{j \neq i} \lambda^2_j (s^*_1 | e^*_1)} \text{ or any probability if } \sqrt{\sum_{j \neq i} \lambda^2_j (s^*_1 | e^*_1)} = a_1 = 0. \]

**Proposition 2** A signaling TI-game where each player makes a single move in a sequence, is equivalent to a classical signaling game for appropriately defined prior beliefs. The eigentypes are identified with the Harsanyi type. The pure equilibrium strategy profile of a signaling TI-game is a Perfect Bayesian equilibrium.

The proof follows from the definition of the signaling TI-equilibrium.

Again we obtain an equivalence result. This should not surprise us because signaling games only involves a single measurement of each player’s type (as in a static TI-game). And we know that within a set of eigentypes \( E_i \), the von Neuman-Luder’s postulate implies Bayesian updating.

### 4.2 Multi-move games with observed actions

In the previous section we noted that commuting multi-stage TI-games without observation are not distinguishable from static TI-game both of which are equivalent to static incomplete information games. This result extends further to commuting dynamic games with observed action.

**Result 3**

Commuting dynamic Ti-games are not distinguishable from classical games in terms of equilibrium predictions.

We postpone the formal proof of this result which follows from a general result proving the equivalence beween the quantum and the classical predictions in a context where all measurements are commuting (see e.g., [7] and [18] for a derivation of this result in a Social Sciences context).

**Non-commuting multi-stage TI-games: strategic manipulation of players’ type**

We are interested in multi-stage game with observed actions where at each period \( t \) the players simultaneously choose their action which are revealed at the end of the period.

We let as before the set of player be \( N = \{1, 2\} \). And we consider a two-stage game \( t = 1, 2 \) where the first stage corresponds to the Game Situation \( G(h^0) \) where \( h^0 \) is the history at date 0. The second stage’s GS is \( G(h^1) \) which does not commute with \( G(h^0) \). Let \( A_j (h^t) \) be a finite set of actions available to player \( j = 1, 2 \) after history \( h^t = (a^0, ..., a^{t-1}) \) where \( a_t = (a^1_t, a^2_t) \) is the vector of date-t actions. A pure behavioral strategy \( s \) maps the set of possible histories \( \mathcal{H}^t \) and types into the action spaces:
where all the strategies yield a Bayes Nash equilibrium not only for the whole game but for all continuation
history \( h \end{period} t \). For simplicity we assume that only the own types affects utility. As we defined
utility, it is the sum of the utility of the profile of date-\( t \) actions evaluated by the resulting types at the
end of period \( t \).

Each player is represented by his type \( |t^0_j \rangle \in T \). As before we have \( E_j ( G ( h^0) ) \subset T \), where we
write \( E_j ( G ( h^0) ) = E_j ( h^0) = \{ e^1_j ( h^0), ..., e^\delta_j ( h^0) \} \) is the set of eigentypes of player \( j \) in \( G(h^0) \). And
\( E_j ( h^1) \subset T \), where \( E_j ( h^1) = \{ e^1_j ( h^1), ..., e^\delta_j ( h^1) \} \) is a set of eigentypes of the GS associated with
history \( h^1 \).

The standard classical equilibrium concept is that of Perfect Bayesian equilibrium. It is an equilibrium
where all the strategies yield a Bayes Nash equilibrium not only for the whole game but for all continuation
game starting in each period after every possible history \( h^t \). To make the continuation games into a true
game one must specify the beliefs at the start of each continuation game. We shall proceed similarly
and characterize the properties of the equilibrium updating rule consistent with the Type Indeterminacy
hypothesis.

The new feature of non-commuting TI-games is that the two consecutive GS are associated with two
distinct sets of eigentypes and that the set \( E ( h^0) \times E ( h^1) \) does not exist. This implies that updating
cannot be purely Bayesian. Instead it will involve two steps and quantum probability calculus. As we
shall see this later feature opens up for strategic "manipulation" of the opponent’s type.

Step 1

Let \( (|t^0_1 \rangle, |t^0_2 \rangle) \) be the initial common knowledge pure type, \( |t^0_1 \rangle = \sum_i \lambda_i |e_i ( h^0) \rangle \) and \( |t^0_2 \rangle = \sum_i \gamma_i |e_i ( h^0) \rangle \). The eigentypes of both players share the initial beliefs

\[ B0 : \mu ( e^1_i ( h^0)) = \lambda^2_i, \mu ( e^2_i ( h^0)) = \gamma^2_i. \]

The first step is as decribed in the static TI-game in sect.5. For initial types \( (|t^0_1 \rangle, |t^0_2 \rangle) \), period
1 actions \( (a^1_1, a^1_2) \) and strategy profile \( (s_1, s_2) \), the von Neuman-Luder projection postulate is used to compute the resulting type vector \( (|t^1_1 ( h^1) \rangle, |t^1_2 ( h^1) \rangle) \) in terms of \( G(h^0) \) eigenvectors:

\[ |t^1_1 ( h^1) \rangle = \sum_i \lambda'_i |e_i ( h^0) \rangle \]  \hspace{1cm} (15)

where the sum is taken over \( i \) such that \( s_1 ( e^1_i ( h^0)) = a^1_1 \) and \( \lambda'_i = \frac{\lambda_i}{\sqrt{\sum_k \lambda_k^2 ( s_1 ( e^1_k ( h^0)) = a^1_1)} } \). The beliefs
in term of the eigentypes of \( G ( h^0) \) are

\[ B1 : \mu ( e^1_i ( h^0)|a^1_1) = \lambda'^2_i. \]

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As we earlier noted this is equivalent to Bayesian updating (for appropriately defined initial beliefs) which is thus applied whenever it is possible.54

Step 2

The second step involves a "translation" of the types resulting from the first actions in eq. (15) so that we express them in terms of the eigentypes of $G\left(h^1\right)$. This operation is necessary to be able to compute the equilibrium strategy in the subgames following $h^1$. Indeed the potential $G\left(h^1\right)$ — eigentypes of each player must reason using their expectation about the opponent’s play. But that expected play is computed from the best-replies of the opponent’s $G\left(h^1\right)$ — eigentypes and from their relative probability weights in the type vector resulting from period 1. The translation is performed using a basis transformation matrix $T_{G\left(h^0\right)\mid G\left(h^1\right)}$ that links the two non-commuting GS.55 We can express each eigentype of $G\left(h^0\right)$ in terms of the eigentypes of $G\left(h^1\right)$:

$|e_i\left(h^1\right)\rangle = \sum_j \delta_{ij} |e_j\left(h^1\right)\rangle$

where $\delta_{ij}$ are the elements of the basis transformation matrix $\delta_{ij} = \langle e_j\left(h^1\right) | e_i\left(h^0\right) \rangle$.56 Collecting the terms we can write

$$|t_1\left(h^1\right)\rangle = \sum_j \left( \sum_i \lambda_i^j \delta_{ij} \right) |e_j\left(h^1\right)\rangle$$

The probability for eigentype $e_i^1\left(h^1\right)$ of player 1 at date $t = 1$ is

$$B2 : \mu\left( e_i^1\left(h^1\right) | a_1 \right) = \left( \sum_i \lambda_i^j \delta_{ij} \right)^2$$

This is the crucial formula that captures all the difference between the classical and the quantum approach. B2 is not a conditional probability formula where the $\delta_{ij}^2$ are statistical correlations between the eigentypes at the two stages. The player is a non-separable system with respect to GS $\left(h^0\right)$ and GS $\left(h^1\right)$.57 As a result, the updated beliefs are given by the square of a sum (implying cross terms) and not the sum of squares.

To see this, consider a simple symmetric example and set $k = 2$, so $E\left(h^0\right) = \{ e_1\left(h^0\right), e_2\left(h^0\right) \}$ and $E\left(h^1\right) = \{ e_1\left(h^1\right), e_2\left(h^1\right) \}$. Similarly, let $A\left(h^0\right) = \{ a_1, a_2 \}$ and $A\left(h^1\right) = \{ b_1, b_2 \}$. Consider a strategy profile such that the eigentypes of player 1 have the same best-reply to the expected play of player 2 i.e., they pool, they both choose $a_1$. In that case there is no updating $|t_1\left(h^0\right)\rangle = |t_1\left(h^1\right)\rangle = \lambda_1 |e_1^1\left(h^0\right)\rangle + \lambda_2 |e_2^1\left(h^0\right)\rangle$. Assume next that at stage 2 $e_1^1\left(h^1\right)$ and $e_2^1\left(h^0\right)$ choose respectively $b_1$ and $b_2$.

54 $\mu\left( e_i\left(h^0\right) | a_1 \right)$ is any probability if $\sqrt{\sum_{j \neq i} e_j\left(h^0\right) | \lambda_j^2 \langle e_i^1\left(h^0\right) | e_j\left(h^0\right) \rangle} = a_1 = 0$.

55 The transformation links the two non-commuting GS. It is independent of the player’s identity.

56 We note that the basis transformation matrix does not depend on the individual players, it is the same for both.

57 This means that $|t_1^2\rangle$ cannot be written as a tensor product type composed of $|t_1^1\rangle_{GS\left(h^0\right)}$ and $|t_1^1\rangle_{GS\left(h^1\right)}$ or equivalently that the tensor product set $E\left(h^0\right) \otimes E\left(h^1\right)$ does not exist.
in response to player 2’s expected play. The ex-ante (at date \( h^0 \)) probability for the play of \( b_1 \) by player 1 is:

\[
\mu (e_1^1 (h^1)) = (\lambda_1 \delta_{11} + \lambda_2 \delta_{21})^2 = \lambda_1^2 \delta_{11}^2 + \lambda_2^2 \delta_{21}^2 + 2 \lambda_1 \lambda_2 \delta_{11} \delta_{21}.
\]

Consider now another strategy profile where player 1’s eigentypes do not have the same best reply to the expected play of player 2 i.e., they separate at stage 1. Then, \( |t_1 \( h^0 \)) \neq |t_1 \( h^1 \)) \). We have \( |t_1 \( h^1 \)) = |e_1 \( h^0 \)) \) with probability \( \lambda_1^2 \) and \( |t_1 \( h^1 \)) = |e_2 \( h^0 \)) \) with probability \( \lambda_2^2 \). We assume for simplicity that at stage 2, the \( h^1 \)-eigentypes separate as before in response to player 2’s expected play.

The ex-ante probability for the play of \( b_1 \) is the probability for each of the possible resulting type times the correlation with the eigentype of \( G (h^1) \) that chooses \( b_1 \):

\[
\mu (e_1^1 (h^1)) = \lambda_1^2 \delta_{11}^2 + \lambda_2^2 \delta_{21}^2.
\]

Hence, although player 1’s strategy at stage 2 does not depend on \( h^1 \): it is simply that eigentype \( e_1^1 (h^1) \) plays \( b_1 \), his actual expected play at stage 2 does! The probability for \( b_1 \) is not the same in (17) and (18) because the type of player 1 has been modified by his best-reply to player 2. Consequently when player 2 always (for all \( h^1 \)) prefers player 1 to play \( b_1 \) he should clearly try to induce player 1’s eigentypes to pool at stage 1 if the term \( 2 \lambda_1^2 \lambda_2 \delta_{11} \delta_{21} \) is positive (and to separate if the term is negative).

Thus, when considering a move a player must account not only for the best-reply of his opponent as usual but also for the induced resulting type of the opponent. More precisely Type Indeterminacy gives a new strategic content to pooling respectively separating moves, a content that goes beyond the informational one. When some eigentypes of a player pool, that player remains indeterminate (superposed) with respect to those eigentypes. this conserved indeterminacy implies that in the next following (non-commuting) GS, these superposed eigentypes may interact with each other producing interference effects that affect the probabilities for the different play. Property B2 of the \( \mu (\cdot) \) function secures that the players takes into account the impact of a play on the resulting profile of types.

For each player \( i \), history \( h^t \), eigentype \( e_j^t (h^t) \) and alternative strategy \( s' \)

\[
P : u_i (s|h^t, e_j^t (h^t), \mu (., h^t)) \geq u_i (s'|h^t, e_j^t (h^t), \mu (., h^t))
\]

**Definition 1** A Perfect TI-equilibrium is a pair \( (s, \mu) \) that satisfies conditions P and B0-B2 above.
Next follows an example showing that a Perfect TI-game equilibrium is not necessarily a Perfect Bayesian equilibrium.

Example

We have 2 players and the following sequence of moves:

- **stage 1** both players choose simultaneously between Left or Right and
- **stage 2** player 1 chooses between Up and Down which ends the game. The payoffs are obtained at the end.

Let \( E(h^0) = \{\theta_1, \theta_2, \theta_3\} \) with \( \theta_1 \) : always prefers to go Left (stubborn or S-type); \( \theta_2 \) : prefers to do the same as the opponent (coordination or C-type) so if e.g., 2 is expected to go Left with 50% or more chance \( \theta_2 \) chooses to go Left and Right if the expectation is less than 50%; \( \theta_3 \): prefers to do the contrary of the opponent (anti-coordination or AC-type) following the same logic in his best response to expected play as \( \theta_2 \).

The initial types of the players are

\[
|t_1\rangle = \sum \lambda_i |\theta_i\rangle \quad \text{and} \quad |t_2\rangle = \sum \gamma_i |\theta_i\rangle.
\]

for concreteness we set \( \lambda_1 = \lambda_2 = \lambda_3 = \sqrt{3/3} \) and \( \gamma_1 \approx 0, \gamma_2 \approx \sqrt{0.4} \) and \( \gamma_3 = \sqrt{3} \).

We first consider a case when stage 1 is played alone. There is a pure strategy equilibrium where the eigentypes of each player play as follows:

- Player 1 : \( \theta_1^1 \) plays \( L \), \( \theta_1^2 \), \( R \), and \( \theta_1^3 \) plays \( L \), which we write \( \{(\theta_1^1, L), (\theta_1^1, R), (\theta_1^3, L)\} \) and for player 2 we have \( \{(\theta_2^1, L), (\theta_2^2, L), (\theta_2^3, R)\} \).

We check that this is an equilibrium. Player 1 plays \( L \) with prob \( \lambda_1^2 + \lambda_3^2 > .5 \) so player 2’s AC-type best-responds with \( R \) with probability \( \gamma_3^2 = .6 > .5 \) so player 1 C-type best-responds with \( R \) with probability \( \lambda_2^2 \) while player 2’s C-type plays \( L \) with probability \( \gamma_2^2 = .4 \). So all eigentypes do play a best reply to the opponent expected play.

Now consider the whole game where both stages are played. We shall assume that player 1’s preference between Up and Down are the same whatever happened before (at stage 1). So from the point of view of player 1, \( GS(h^1) \) is the same for all \( h^1 \) and it is a simple decision situation \( DS \). Player 1’s preferences and play does not depend on player 2. We assume that the \( DS(GS(h^1)) \) is an operator that does not commute with the \( GS(h^0) \) which means that the eigentype \( \theta_i \) can be expressed in terms of the eigentypes of the \( DS \):

\[
|\theta_1\rangle = \alpha_1 |U\rangle + \alpha_2 |D\rangle; \quad |\theta_2\rangle = \beta_1 |U\rangle + \beta_2 |D\rangle; \quad |\theta_3\rangle = \delta_1 |U\rangle + \delta_2 |D\rangle
\]  

(19)
with $\delta_1$ close to 0 to simplify the presentation.

Assume further that Player 2 (end)payoff depends critically on player 1’s stage 2 decision: his payoff of Up for any path of stage 1 is between 100 and 150 and his payoff of D is 0 for all paths of stage 1.

We next show that a Bayesian equilibrium of the classical version of the game is no longer an equilibrium in the TI-game version. In a classical version of the game player 1’s type characteristics (preferences) are fixed, they are given by the initial move by Nature. As usual we take the square of the coefficients of superposition to describe the corresponding classical incomplete information model. And we obtain that the ex-ante probability for $U$ is $\text{Prob}(U) = \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2$. It does not depend on the play at stage 1 so the equilibrium depicted above is part of a Bayesian equilibrium for the whole game.

What about the TI-model? First, we show that with the equilibrium of stage 1 depicted above we obtain the same outcome as in the classical case. We note that the equilibrium depicted above Player 1’s resulting type is $|t_0^i| = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} |\theta_1| + \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} |\theta_2|$ with probability $\lambda_1^2 + \lambda_2^2$, and $|t_0^i| = |\theta_3|$ with probability $\lambda_3^2$. Using the expressions in (19) the ex-ante probability for Up equal to

$$(\lambda_1^2 + \lambda_3^2) \left( \frac{\lambda_1^2}{\lambda_1^2 + \lambda_3^2} \alpha_1^2 + \frac{\lambda_2^2}{\lambda_1^2 + \lambda_3^2} \beta_1^2 \right) = \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2.$$

But we claim that under some condition namely if $2\lambda_1 \alpha_1 \lambda_2 \beta_1 > 0$ this is no longer an equilibrium of the TI-game. Player 2 can do better. He can increase the probability for the play of U when he realizes that his play at stage 1 influences the type of his opponent (and thus his subsequent expected play). More specifically player 2’s $\theta_3^2$ eigentype realizes that if he plays L then the $\theta_3^2$ of player 1 will choose L and will pool with $\theta_1^1$ ($\theta_3^3$ choose R). The resulting type of player 1 is then

$|t_0^i| = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\theta_1| + \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} |\theta_2|$ with probability $\lambda_1^2 + \lambda_2^2$ and $\theta_3$ with probability $\lambda_3^2$. This implies an ex-ante probability for the play of U equal to (recall $\theta_3$ never plays U)

$$\text{prob}(U; |t_0^i|) = \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2 + 2\lambda_1 \alpha_1 \lambda_2 \beta_1 > \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2.$$

In the classical Bayes-Harsanyi version of this game, player 2 has no means of influencing player 1’s play at stage 2. This play depends exclusively on the type of player 1, which is known to player 1 from the beginning of the game.

In the TI-version of this game, the expected move of player 2 at stage 1 induces a measurement of player 1’s indeterminate type. Different expected move induces different measurement because the eigentypes of player 1 best-replies to the expected play. The impact of the measurement on the type of player 1 is to induce some patterns of separation( or pooling) between eigentypes. In our example
when indeterminacy is preserved (by pooling) between $\theta_1, \theta_1^2$, this gives rise to interference effects in the determination of the probability for $Up$. By construction player 2 strictly (whatever his eigentype) prefers player 1 to play $U$. So player 2 optimal strategy boils down to maximizing the probability for the play of $U$. In our example $\theta_1$ and $\theta_2$ can interfere positively to increase the probability for the play of $U$ while $\theta_1^3, \theta_2^3$, and $\theta_3^3$ cannot. Hence, the optimal strategy of player 2 is to induce $\theta_1$ and $\theta_2$ to pool at stage 1. This is achieved by the proposed first stage strategy profile $\{(\theta_1^1, L), (\theta_2^1, L), (\theta_1^3, R)\}$ and $\{(\theta_1^2, L), (\theta_2^2, L), (\theta_3^2, L)\}$ with the second stage play being computed mechanically and this is the only Perfect TI-equilibrium of this game.

We note that the proposed strategy at stage 1 $\{(\theta_1^1, L), (\theta_2^1, L), (\theta_1^3, R)\}$ and $\{(\theta_1^2, L), (\theta_2^2, L), (\theta_3^2, L)\}$ are not part of a Bayesian equilibrium of the whole game.

**Theorem 1** *Characterize the conditions under which the set of Perfect Bayesian equilibrium and Perfect TI-equilibrium coincide and under which they do not coincide.*

In progress.

We have identified the distinction between the Bayes Harsanyi and the TI-game approaches in the opportunity to influence upon the resulting type profile. This influence takes the particular form of inducing a particular pattern of separation and pooling of the currently relevant eigentypes.

### 4.3 Possible fields of application of TI-games

We have learned that TI-games may bring forth new results in the context of multi-stage game or when a game is preceded by some form of "pre-play". We conjecture that the Type Indeterminacy approach may bring new light on the following issues:

- Players’ choice of selection principle in multiple equilibria situation;

  In Camerer ?? the author reports about experiments where a pre-play auction impacts on the principle of selection among multiple equilibria in a coordination game. The pre-play auction for the right to play the coordination game tended to push toward the payoff-dominant equilibrium compared with the no pre-play case. In a TI-game, preferences with respect to the equilibrium selection criteria can be modified by pre-play.

- The selection of a reference point;

  According to experiments (see ) playing a contest before an ultimatum game can affect the equilibrium offer and acceptance threshold. In a TI-game the pre-play of a contest may change the preferences of the players with respect to what they feel entitled to in an ultimatum game played next.

- The sunk cost fallacy;

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According to numerous experiments and casual evidence people seem to be the victims of the sunk cost fallacy. In an experiment, people who were offered a year subscription to the theater showed (on average) a greater propensity to go to the theater than people who were not offered subscription. In a TI-game the pre-play decision to purchase a subscription may modify people’s valuation of theater plays.

- Path-dependency;

A single (little probable) move which radically modifies the type of a player can yield significative implications for the path of future play.

We believe that this avenue of research has a rich potential to explain a variety of puzzles in (sequential) interactive situations and to give new impulses to game theory.
References


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